Linear Riley Equilibria in Quadratic Signaling Games[*]

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Abstract

We study signaling games with quadratic payoffs. As signaling games admit multiple separating equilibria, many equilibrium selection rules are proposed and a well-known solution is Riley equilibria. They are separating equilibria in which the sender achieves the highest equilibrium payoff for all types among all separating equilibria. We analyze the conditions for Riley equilibria to be linear, a common assumption in many applications. We derive a sufficient and necessary condition for the existence and uniqueness of linear Riley equilibria. We apply the result to confirm the dominance of linear equilibria in some classic examples, and we show that, in some other examples, there exist previously unknown nonlinear Riley equilibria.

Keywords: Signaling; Riley Equilibria; Linear Strategy

JEL Codes: C73, D82, D83

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1   Introduction

Signaling games play an important role in many areas of social sciences, providing insights into issues such as education (Spence, 1973), limit pricing (Milgrom and Roberts, 1982), leadership (Hermalin, 1998), and insurance (Rothschild and Stiglitz, 1976). In signaling games, a privately informed sender strategically takes an action to influence an uninformed receiver. A particularly important case is when the sender's equilibrium strategy is separating and completely reveals the state.

As signaling games admit multiple separating equilibria, many equilibrium selection rules are proposed and a well-known solution is Riley equilibria (Riley, 1975, 1979). They are separating equilibria in which the sender achieves the highest equilibrium payoff for all types among all separating equilibria. Riley equilibria require the least amount of inefficient signaling and therefore Pareto-dominate all separating equilibria. On the other hand, linear equilibria are widely studied due to their simplicity.[1] This paper answers an important query: does a linear Riley equilibrium exist, and if so, under what conditions?

As linear solutions most commonly appear in quadratic settings, we focus on signaling games with quadratic payoffs. A sender (he) is privately informed about an underlying state $\theta \in \Theta$ and takes an action $a_1 \in \mathbb{R}$. After observing a_1, a receiver (she) chooses an action $a_2 \in \mathbb{R}$. The sender's ideal points for these two actions are linear in the state. The receiver wants her action to match the state. This setup is widely considered in the literature and nests a broad class of models in industrial organization and organizational economics as shown in Section 5.

The analysis of quadratic signaling games is technically challenging, as the widely adopted assumption of belief monotonicity and the single-crossing condition (for example, Spence, 1973; Mailath, 1987; Roddie, 2011; Mailath and von Thadden, 2013) do not necessarily hold.[2] Without these assumptions, our analysis requires novel arguments for analyzing Riley equilibria.

Our main result provides a necessary and sufficient condition for the existence and uniqueness of linear Riley equilibria (Theorem 1). Our results imply that the common restriction to linear strategies is only partially justified and that in general a wider class of strategies is possible. To guarantee that Riley equilibrium is linear, one needs

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[1]For example, Bonatti and Cisternas (2019); Argenziano and Bonatti (2021); Ball (2021) study linear equilibria in signaling games with quadratic payoffs.

[2]See Section 6 for a more detailed discussion.

to impose restrictions not only on the preference parameters but also on the state space. On the other hand, we establish equilibrium properties such as continuity, differentiability, and monotonicity (Theorem 2).

We apply our results to the following four examples in the literature. The classic model of leadership in Hermalin (1998) studies how a leader incentivizes employees to exert effort on a project. The leader privately observes the valuation of the project and publicly exerts effort. Based on the leader's effort, the other workers make inferences about the project's valuation and exert effort simultaneously. We show that whether Riley equilibrium is linear crucially depends on the state space. Hermalin (1998) focuses only on linear equilibria. Yet, if we slightly perturb the space of the valuation, Riley equilibrium is no longer linear.

Argenziano and Bonatti (2021) consider a dynamic model of behavior-based price discrimination. A consumer sequentially interacts with two firms. In the first period, firm 1 sets a price and a quality level and the consumer chooses the quantity to consume. A data linkage allows the second firm to observe the first-period outcome. After observing the outcome, the second firm tailors its quality level and price to the consumer's type. Argenziano and Bonatti (2021) focus on linear Bayesian Nash equilibria. We show that there exist previously unknown nonlinear Riley equilibria under alternative parameters.

Kartik et al. (2007) analyze a model of strategic communication between an informed but upwardly biased sender and an uninformed receiver. The sender bears a cost of lying about his private information. Kartik et al. (2007) show that the sender's message is biased above the state. Dispensing with the belief monotonicity assumption allows us to consider a case in which the sender's preference is biased upward for some states but downward for other states. We generalize Kartik et al.'s (2007) result by showing that the language is inflated if and only if the sender is upwardly biased (Proposition 3).

Following Aghion et al. (2004), we study a delegation problem where a principal faces an informed but biased agent. The principal delegates control to the agent to use his local knowledge. After learning from the agent's decision, the principal reclaims control and make decisions by herself. In this example, we show that Riley equilibrium is also optimal for the principal (Proposition 4).

Literature Review

We contribute to the literature on Riley equilibrium by studying its linearity. Riley equilibrium is important in its own right and the literature has shown that it admits many equilibrium refinements. Cho and Kreps (1987) show that in Spence's model of job-market signaling with two types, the only equilibrium not rejected by the intuitive criterion is Riley equilibrium. Even with more than two types, Cho and Sobel (1990) show that a stronger criterion, called the D1 refinement, selects Riley equilibrium. Ramey (1996) shows that with multiple signals and a continuum of states, the D1 refinement selects Riley equilibrium under the Spence–Mirrlees single-crossing condition.[3],[4]

Our paper contributes to the literature on signaling games by dispensing with several common assumptions, including belief monotonicity and the single-crossing condition (for example, Spence, 1973; Mailath, 1987; Roddie, 2011; Mailath and von Thadden, 2013).[5] Mailath (1987) studies signaling games with a continuum of types and introduces belief monotonicity to discuss the differentiability of separating strategies. In that paper, belief monotonicity is needed to provide two additional conditions. The first is an initial value condition. The second is the Spence–Mirrlees single-crossing condition. Each of these conditions (combined with regularity conditions) implies differentiability of separating strategies.

Mailath (1987) pins down Riley equilibrium with the initial value condition. That is, the worst type takes his most preferred action (with no inefficient signaling). He mentions that "the worst type is the worst belief off the equilbrium path. A deviation by the worst type to his most preferred action cannot be credibly punished." This logic is exactly the same as checking whether an equilibrium payoff profile fails the intuitive criterion and D1. We generalize the initial value condition in Section 6.4.

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[3]Similarly, Nöldeke and Samuelson (1997) consider a dynamic Spencian model with perturbations and show that the only separating equilibrium selected by this model is Riley equilibrium.

[4]Both the intuitive criterion and divinity criterion rely on forward-induction arguments. A growing body of work has given precise foundations for solution concepts based on forward-induction, such as extensive-form rationalizability, in terms of assumptions of strong belief in rationality (see Battigalli and Siniscalchi, 2002; Battigalli, 2006; Battigalli and Prestipino, 2013; Battigalli and Catonini, 2021).

[5]In the literature, there are some other assumptions that improves the tractability of signaling games. For example, Kartik et al. (2007) propose a *direction condition*: given a correct belief, taking a higher action and inducing a higher belief affect the sender's payoff in the same direction. In the same vein, Mailath and von Thadden (2013) assume *action monotonicity*: under a correct belief, the sender always prefers a higher action. Notice that none of these assumptions necessarily hold in our model.

2   The Model

We study games with quadratic payoffs. Let $\theta \in \Theta$ denote the state of the world. The state space $\Theta = [m, M]$ is a bounded closed interval in $\mathbb{R}$. A sender (he) is privately informed about the underlying state θ and takes an action $a_1 \in \mathbb{R}$. After observing a_1, a receiver (she) chooses an action $a_2 \in \mathbb{R}$. The receiver's payoff is given by $u^R(\theta, a_2, a_1)$. The sender's payoff is given by

$$U^S(\theta, a_2, a_1) = -(a_1 - a_1^S(\theta))^2 - \delta(a_2 - a_2^S(\theta))^2,$$

where $a_1^S \colon \Theta \to \mathbb{R}$ and $a_2^S \colon \Theta \to \mathbb{R}$ denote the sender's most preferred actions for a_1 and a_2, respectively. The parameter $\delta \geq 0$ captures the sensitivity of the sender's payoff to the receiver's action.[6] The quadratic form is tractable and allows for closed-form solutions.

We assume that the receiver's payoff $u^R(\theta, \cdot, a_1)$ is uniquely maximized at $a_2 = a^R(\theta)$, which is independent of the sender's action. In addition, we assume that $a^R(\theta)$ is continuous and strictly increasing in θ. We normalize[7] $a^R(\theta) = \theta$. Moreover, we assume that the sender's ideal points a_1^S and a_2^S are linear in the state: $a_1^S(\theta) = k_1\theta + b_1$ and $a_2^S(\theta) = k_2\theta + b_2$ with $k_1 \neq 0$. We discuss applications of quadratic signaling games in Section 5.

Our interest is in *separating* equilibria, i.e., equilibria in which the sender plays different actions in different states. A pure separating strategy for the sender is a one-to-one mapping $\sigma \colon \Theta \to \mathbb{R}$. Let $\mu \colon \mathbb{R} \to \mathbb{P}(\Theta)$ denote the receiver's belief over the states after observing the sender's action, where $\mathbb{P}(\Theta)$ denotes the set of probability distributions on Θ. Given a strategy σ, let $A_1(\sigma) \triangleq \{\sigma(\theta) | \theta \in \Theta\}$ be the range of strategy σ. Bayes' rule requires that if σ is separating, for any $a \in A_1(\sigma)$, $\mu(a)$ is a point-mass distribution (Dirac function) on $\sigma^{-1}(a)$. For any $a \in \mathbb{R} \setminus A_1(\sigma)$, all beliefs are permissible, but we can show that it is without loss of generality to restrict attention to point-mass beliefs ($\mu \colon \mathbb{R} \to \Theta$). Let $V(\theta, \hat{\theta}, a_1)$ denote the sender's reduced-form payoff from taking action a_1 when the true state is θ and the receiver

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[6]Our results apply to the case where $\delta < 0$ as well. But we need to additionally check a second-order condition (3). We provide an example in Section 5.2.

[7]Indeed, we can relabel the state to be equal to $\tilde{\theta} \triangleq a^R(\theta)$.

infers $\hat{\theta}$ and best responds to this belief, i.e.,

$$V(\theta, \hat{\theta}, a_1) = -(a_1 - a_1^S(\theta))^2 - \delta(\hat{\theta} - a_2^S(\theta))^2. \tag{1}$$

A *separating PBE* is a one-to-one strategy, $\sigma \colon \Theta \to \mathbb{R}$, and a receiver's belief, $\mu \colon \mathbb{R} \to \Theta$, such that the following conditions hold:

1. Belief consistency: $\forall \theta \in \Theta$, $\mu(\sigma(\theta)) = \theta$,

2. Incentive compatibility: $\forall \theta, \theta' \in \Theta$,

$$V(\theta, \theta, \sigma(\theta)) \geq V(\theta, \theta', \sigma(\theta')),$$

3. Off-path belief: $\forall \theta \in \Theta$, $\forall a \notin A_1(\sigma)$, $V(\theta, \theta, \sigma(\theta)) \geq V(\theta, \mu(a), a)$.

As there may be many separating equilibria, we rank them in terms of the sender's payoff. A separating equilibrium is the *Riley* equilibrium if it gives the sender the highest payoff among all separating PBE at every state. Formally, given an equilibrium where the sender plays strategy σ, this equilibrium is Riley equilibrium if, for all $\theta \in \Theta$,

$$V(\theta, \theta, \sigma(\theta)) \geq V(\theta, \theta, \sigma'(\theta))$$

holds for all σ' played in any separating equilibrium.

In equilibrium, the receiver takes her most preferred action $a^R(\theta) = \theta$, which might differ from the sender's ideal point $a_2^S(\theta) = k_2\theta + b_2$. The distance between them is $a_2^S(\theta) - a^R(\theta) = k_2\theta + b_2 - \theta$. First, we define the *preference-aligned state* θ_0 in which this distance is zero:

$$\theta_0 = k_2\theta_0 + b_2.$$

That is, $\theta_0 \triangleq \frac{b_2}{1-k_2}$, where θ_0 may or may not be in Θ, depending on the values of the parameters.[8] Second, this distance as a function of the state θ increases at rate (k_2-1). The larger the distance, the smaller the sender's payoff. Recall that δ captures the sensitivity of the sender's payoff to the receiver's action. To measure the marginal benefit of inducing a higher belief, we define the *marginal impact* $r \triangleq \delta(k_2 - 1)$.

We now provide an overview of our main result.

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[8]If $k_2 = 1$, we can define $\theta_0 = \infty$.

Theorem 1. *There exists a linear Riley equilibrium if and only if $r > 0$ and $\theta_0 \in \Theta$. Moreover, whenever it exists, it is unique.*

We provide the analytical solution of the linear Riley equilibrium in Appendix A. Proving this result requires several steps of analysis. Specifically, it follows directly from our Proposition 2 and Theorem 2.

3 Preliminary Analysis

In this section, we characterize incentive-compatible separating strategies. We first introduce a useful definition. Let $S \colon \Theta \to \mathbb{R}$, $S(\theta) \triangleq \sigma(\theta) - a_1^S(\theta)$ denote the *strategic distortion*—that is, the distance between the sender's action and his ideal point. If the sender is completely myopic, he ignores the inferential impact of his action and chooses $S(\theta) = 0$ for all θ. By contrast, the strategic sender distorts his action away from the myopic benchmark. Consequently, $S(\theta)$ captures the strategic distortion. From now on, we focus on S, as solving for S is equivalent to solving for the sender's strategy.

In the following analysis, we first identify some necessary conditions of incentive compatibility—that is, what properties a separating strategy must have to be incentive compatible. Then we combine these necessary conditions to make them jointly sufficient.

First, notice that the sender's equilibrium payoff must be continuous. Otherwise, at the discontinuity, incentive compatibility fails. Since the sender's equilibrium payoff is $-S^2(\theta) - \delta(\theta - a_2^S(\theta))^2$, the function $|S|$ must be continuous everywhere.

Second, at any state where the sender's strategy is continuous, how S varies with the state is restricted by incentive compatibility. We define the sender's payoff in state θ when he takes action $\sigma(\theta')$ and induces belief θ' as

$$\mathscr{U}(\theta'; \theta) \triangleq V(\theta, \theta', \sigma(\theta')),$$

where we suppress the dependence on σ. Incentive compatibility is equivalent to

$$\theta \in \arg\max_{\hat{\theta}} \mathscr{U}(\hat{\theta}; \theta).$$

If the sender's strategy is differentiable, two necessary conditions for the maximization

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problem described above must be satisfied. The first one is its first-order condition:

$$\left.\frac{\partial \mathscr{U}(\hat{\theta};\theta)}{\partial \hat{\theta}}\right|_{\hat{\theta}=\theta} = 0, \quad \forall \theta \in (m, M),$$

which we can write explicitly as a differential equation of S:

$$S(\theta)[\frac{dS(\theta)}{d\theta} + k_1] = r\theta + \delta b_2. \tag{2}$$

The second necessary condition is its second-order condition:

$$\left.\frac{\partial^2 \mathscr{U}(\hat{\theta};\theta)}{\partial \hat{\theta}^2}\right|_{\hat{\theta}=\theta} \leq 0, \quad \forall \theta \in (m, M). \tag{3}$$

Third, when an incentive-compatible separating strategy is discontinuous, the direction of the jump is controlled by incentive compatibility. We can show that the direction has the same sign as k_1.[9] In the following proposition, we summarize the above discussions by three necessary conditions. It turns out that they are jointly sufficient. We thus obtain a tractable characterization of incentive compatibility.

Proposition 1 (Characterization). *A separating strategy S is incentive compatible if and only if all of the following conditions hold:*

1. *$|S|$ is continuous.*

2. *At every point in (m, M) where S is continuous, S is differentiable and satisfies the first-order condition (2) and the second-order condition (3).*

3. *If S is discontinuous, then, at each discontinuity, both left and right limits exist and the jump has the same sign as k_1.*

Incentive compatibility is a global concept that requires $\mathscr{U}(\theta;\theta) \geq \mathscr{U}(\hat{\theta};\theta)$ for all $\theta, \hat{\theta} \in \Theta$. Yet using this requirement to check incentive compatibility is tedious. The proposition improves the tractability of identifying Riley equilibria. We use this result to solve for Riley equilibria in the next section.

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[9] An incentive-compatible separating strategy need not be either monotonic or continuous. We provide an example in the proof of Proposition 4.

4 Main Results

In this section, we characterize Riley equilibria that maximize the sender's payoff in all states. One question of particular interest is whether and when Riley equilibria are linear. Before answering this question, we first need to know when a linear incentive-compatible strategy exists. Hence, we first explore its existence.

4.1 Existence of Linear Incentive-Compatible Strategies

It turns out a linear incentive-compatible strategy does not always exist. We show that its existence is determined by what we call the *discriminant* of the game: $\Delta \triangleq k_1^2 + 4r$, where $r = \delta(k_2 - 1)$.

Proposition 2 (Discriminant). *There exists a linear incentive-compatible separating strategy if and only if $\Delta \geq 0$. Moreover, it takes the form $S(\theta) = t(\theta - \theta_0)$, where t is the slope.*

Proof of Proposition 2. Necessity: Fix some linear strategy $S(\theta) = t(\theta - \theta_0) + l$, parameterized by t and l. Then the marginal value of inducing a higher belief is

$$\left. \frac{\partial \mathscr{U}(\hat{\theta}; \theta)}{\partial \hat{\theta}} \right|_{\hat{\theta}=\theta} = -2[(t^2 + k_1 t - r)(\theta - \theta_0) + l(t + k_1)].$$

For S to be incentive compatible, $\left. \frac{\partial \mathscr{U}(\hat{\theta};\theta)}{\partial \hat{\theta}} \right|_{\hat{\theta}=\theta}$ must vanish for every θ. In particular, the coefficient of θ must vanish, i.e., $t^2 + k_1 t - r = 0$. The discriminant of this equation for t is exactly Δ, which is the discriminant of the game. When Δ is negative, there are no solutions, and it follows that $\left. \frac{\partial \mathscr{U}(\hat{\theta};\theta)}{\partial \hat{\theta}} \right|_{\hat{\theta}=\theta}$ does not vanish for all θ. Thus, no linear strategy can be incentive compatible.

Sufficiency: When $\Delta \geq 0$, differential equation (2) has linear solutions (see Appendix A). Moreover, they satisfy the second-order condition. As linear solutions are continuous, by the Characterization Proposition 1, the linear solutions are incentive compatible.

Now suppose that there is a linear incentive-compatible strategy. By the Characterization Proposition 1, it satisfies differential equation (2). Its linear solutions take the form $S(\theta) = t(\theta - \theta_0)$ (see Appendix A). $\square$

The above proposition implies that the existence of a linear incentive-compatible separating strategy crucially depends on the sign of the discriminant Δ. To illustrate the intuition, we analyze the sender's incentive to manipulate the receiver's belief, which we call the *belief-manipulation incentive*. Given a correct belief θ, the receiver's action leads to the sender's payoff $-\delta(\theta - a_2^S(\theta))^2$. When $k_2 > 1$, $a_2^S(\theta) - \theta$ is positive for $\theta > \theta_0$ (negative for $\theta < \theta_0$), where θ_0 is the preference-aligned state. This gives the sender an incentive to induce a belief slightly farther away from θ_0 because $|a_2^S(\theta) - \hat{\theta}|$ is smaller when inducing a belief slightly farther away from θ_0. Similarly, when $k_2 < 1$, the sender has an incentive to induce a belief slightly closer to θ_0. A incentive-compatible separating strategy has to balance the marginal benefit of belief manipulation with the marginal cost of strategic distortion.

First, we explain why there cannot exist a linear incentive-compatible separating strategy when $r = \delta(k_2 - 1)$ is so negative that $\Delta < 0$. This corresponds to the situation in which $k_2 < 1$ and the two players' preferences are sufficiently misaligned. In the left panel of Figure 1, we plot both players' ideal points in such a situation. Suppose that $k_1 > 0$. To counterbalance the incentive to induce a belief closer to θ_0, we have to set $t \in (-k_1, 0)$. That is, the sender's strategy σ rotates clockwise moderately (around θ_0) relative to $a_1^S(\theta)$ (see the right panel of Figure 1). As the preference gap becomes sufficiently large because of a lower k_2, the belief-manipulation incentive becomes so strong that it outweighs the cost of strategic distortion, no matter what linear strategy the sender uses.

Formally, consider a state $\theta < \theta_0$. In a candidate linear separating equilibrium, the marginal benefit of inducing a belief closer to θ_0 is

$$\frac{\partial}{\partial \hat{\theta}}[-\delta(\hat{\theta} - k_2\theta - b_2)^2]\Big|_{\hat{\theta}=\theta} = -2r(\theta_0 - \theta),$$

whose magnitude is scaled by the marginal impact r. For any linear strategy with slope $t \in (-k_1, 0)$, the marginal cost of inducing a belief closer to θ_0 is

$$\frac{\partial}{\partial \hat{\theta}}[(\sigma(\hat{\theta}) - a_1^S(\theta))^2]\Big|_{\hat{\theta}=\theta} = -2(t^2 + k_1 t)(\theta_0 - \theta),$$

which is maximized at $t = -k_1/2$. When r is so negative that $\Delta = k_1^2 + 4r < 0$, the marginal benefit always outweighs the marginal cost for any $t \in (-k_1, 0)$. Thus, no linear strategy can be incentive compatible.
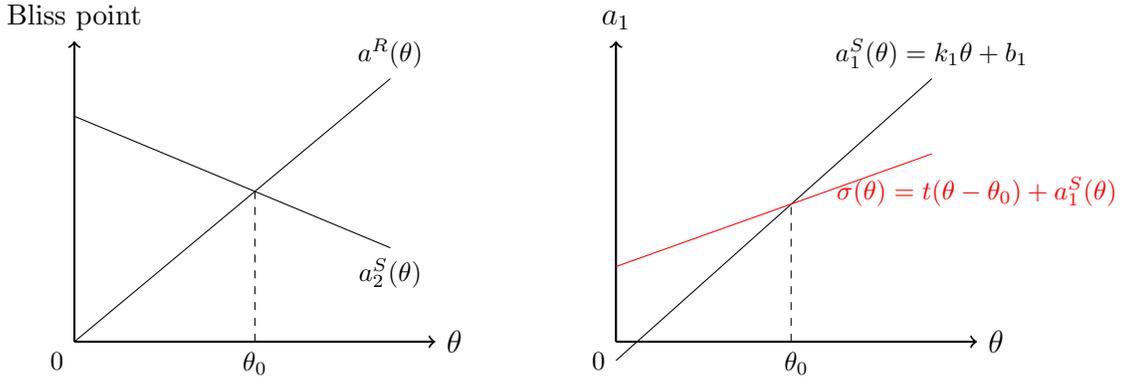
Figure 1: Bliss Point (Left) and Sender's Strategy (Right)

Second, when $r > 0$, we can similarly consider a state $\theta > \theta_0$. The marginal benefit of inducing a belief farther away from θ_0 is $2r(\theta - \theta_0)$ while the marginal cost is $2(t^2 + k_1 t)(\theta - \theta_0)$. We have to set $t \notin [-k_1, 0]$ such that the marginal cost can counterbalance the marginal benefit. As the range of t is unbounded in this case, we can always find a suitable t such that the marginal cost cancels out the marginal benefit. Therefore, an incentive-compatible linear strategy always exists.

Finally, the above analysis also highlights how k_1 determines the existence of a linear incentive-compatible separating strategy. Consider the case with $r < 0$ and $k_1 > 0$ (in Figure 1). To counterbalance the incentive to induce a belief closer to θ_0, we require $t \in (-k_1, 0)$.[10] Then, the strategic distortion $|S|$ increases with k_1. This implies that the marginal cost of inducing a belief closer to θ_0 increases with k_1. When k_1 is sufficiently large, this marginal cost can be strong enough to counterbalance the belief-manipulation incentive for some suitable t. Thus, some linear strategies can be incentive compatible.

The Discriminant Proposition 2 provides a necessary condition for the existence of linear Riley equilibria. For the rest of the paper, we focus on non-negative discriminant games. We relegate the discussion of negative discriminant games to Online Appendix E.

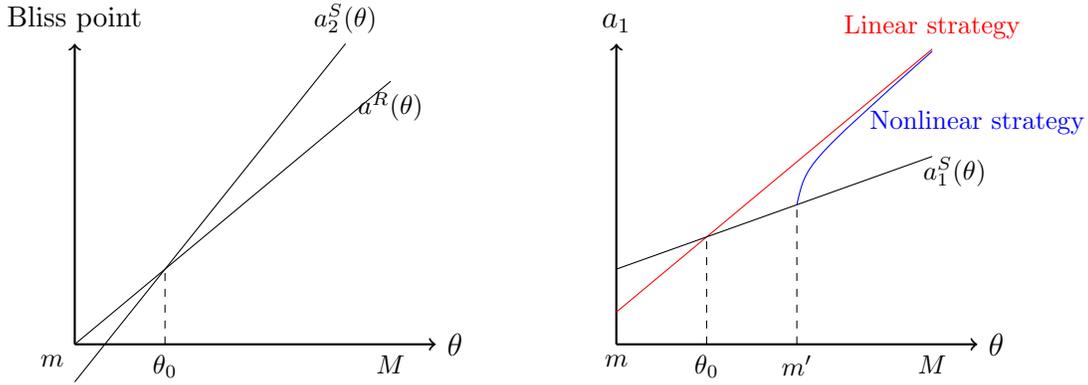[10]The case where $k_1 < 0$ is symmetric. Then it must be true that $t \in (0, -k_1)$.

Figure 2: Bliss Point (Left) and Sender's Strategy (Right) for $r > 0$

4.2 Riley Equilibria

We next solve for Riley equilibria. First, the Characterization Proposition 1 identifies the whole class of IC separating strategies. Second, we pin down the IC separating strategy that maximizes the sender's payoff at every state. Third, we construct a PBE in which this optimal IC separating strategy is played. As the separating PBE is a more demanding notion than incentive compatibility, the PBE constructed must be the dominant separating PBE.

Theorem 2. *In non-negative discriminant quadratic games, there exists a unique Riley equilibrium. In this equilibrium, the sender's strategy is*

1. *continuous and monotonic,*

2. *differentiable on (m, M),*

3. *linear if and only if $r > 0$ and $\theta_0 \in \Theta$.*

We fully solve for Riley equilibrium strategy's closed form in Appendix C. Here we illustrate the intuition for the sufficient and necessary condition of linearity. Recall that the linear IC strategy must cross $(\theta_0, 0)$ by the Discriminant Proposition. When $\theta_0 \notin \Theta$, the strategic distortion of the linear IC strategy is nonzero on Θ. Therefore, we can always find some other solution to equation (2) with a uniformly smaller strategic distortion once we carefully set the initial value to be $S(m) = 0$ or $S(M) = 0$. Now suppose that $\theta_0 \in \Theta$. The more interesting question is why the condition for linearity also requires $r > 0$.
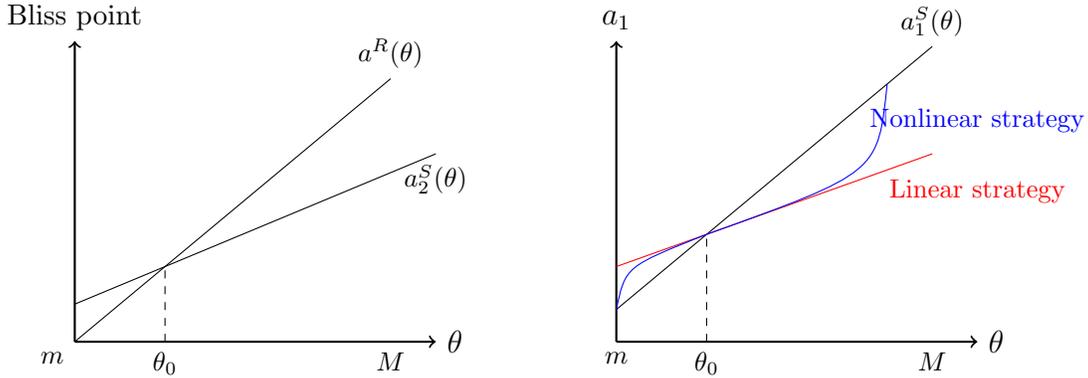
Figure 3: Bliss Point (Left) and Sender's Strategy (Right) for $r < 0$

Consider the case of $r > 0$; that is, $k_2 > 1$. To counterbalance the belief-manipulation incentive, the strategic distortion of an IC separating strategy must deter the sender from inducing a belief farther away from θ_0. Let us consider a state $\theta > \theta_0$. As the belief-manipulation incentive

$$\frac{\partial}{\partial \hat{\theta}} [-\delta(\hat{\theta} - k_2 \theta - b_2)^2]\Big|_{\hat{\theta}=\theta}$$

depends only on the state, the marginal cost of of inducing a belief farther away from θ_0,

$$\frac{\partial}{\partial \hat{\theta}} [(\sigma(\hat{\theta}) - a_1^S(\theta))^2]\Big|_{\hat{\theta}=\theta},$$

must be the same for all IC strategies. This implies that if an IC strategy induces a smaller strategic distortion $|\sigma(\theta) - a_1^S(\theta)|$, it must have a larger derivative. This is shown in Figure 2, in which we plot the linear IC strategy and a nonlinear strategy with a smaller strategic distortion.[11] As $\theta_0 \in \Theta$, the nonlinear strategy that has a larger derivative cannot be extended with full support Θ. This in turn implies that when $\theta_0 \in \Theta$, all supported nonlinear IC strategies are farther from the bliss point a_1^S compared to the linear IC strategy and hence are uniformly dominated.

In the case of $r < 0$—that is, $k_2 < 1$—the strategic distortion of an IC separating strategy must deter the belief-manipulation incentive. In Figure 3, we plot the linear IC strategy and a nonlinear IC strategy with a smaller strategic distortion. Suppose that $\theta > \theta_0$. As the strategic distortion is smaller in the nonlinear IC strategy, as θ

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[11]Notice that if $\theta_0 \notin \Theta$ and Θ is bounded below by $m' > \theta_0$, the nonlinear strategy is indeed an IC separating strategy that uniformly dominates the linear IC strategy.

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decreases, the nonlinear IC strategy must diverge from the bliss point faster and tends to the linear strategy. Thus, we can have a nonlinear IC strategy that dominates the linear strategy.

Remark 1. *In Online Appendix E, we study negative discriminant quadratic games. We find that when $\theta_0 \notin \Theta$, there exists a unique Riley equilibrium where the sender's strategy is nonlinear, continuous, monotonic, and differentiable. And when $\theta_0 \in \Theta$, there is no continuous incentive-compatible separating strategy and, moreover, a separating PBE might not exist.*

5   Applications

Quadratic signaling games include the game studied in Argenziano and Bonatti (2021) and the examples in Kartik et al. (2007). With slight mathematical manipulation, they also include the game in Hermalin (1998). In this section, we apply our results to investigate examples in the literature.

5.1   Leading by Example

The classic model of leadership in Hermalin (1998) studies how a leader incentivizes employees to exert effort on a project. Since the firm benefits from all effort, the leader is motivated to tell employees that all projects deserve their maximum effort. Consequently, rational employees disregard the leader's call. Nevertheless, the leader herself can exert high effort and thereby incentivize her followers to do the same.

In the model, a team contains N identical workers, including a leader. Each worker n exerts effort e_n towards a common endeavor. The value of the common endeavor is $\mathscr{V} = \theta \sum_{n=1}^{N} e_n$, where $\theta \in \Theta = [0, 1]$ denotes a random productivity factor.

A worker's utility is $s_w \times \mathscr{V} - \frac{1}{2}e^2$, where s_w denotes the worker's share, $s_w \times \mathscr{V}$ denotes his wage, and $\frac{1}{2}e^2$ is the disutility from exerting effort. The leader's utility is $s_l \times \mathscr{V} - \frac{1}{2}e^2$, where $s_l + (N-1)s_w = 1$. The leader privately observes θ and publicly exerts effort. The other workers make inferences concerning θ based on the leader's effort. Let $e(\theta)$ denote the leader's strategy in equilibrium, and $\hat{\theta}$ denote the

followers' belief. Each worker n solves the following problem:

$$\max_{e_n} \left[s_w \hat{\theta}(e_n + \sum_{j \neq n} e_j) - \frac{e_n^2}{2} \right].$$

They do so by choosing $e_n = s_w \hat{\theta}$. Therefore, the leader's payoff from exerting effort e when the state is θ and the followers infer $\hat{\theta}$ is given as follows:

$$V(\theta, \hat{\theta}, e) = s_l \theta(e + (N-1)s_w \hat{\theta}) - \frac{e^2}{2}.$$

The leader solves

$$\max_{e(\hat{\theta})} \left[s_l \theta(e(\hat{\theta}) + (N-1)s_w \hat{\theta}) - \frac{e^2(\hat{\theta})}{2} \right]$$

with the first-order condition

$$e'(\theta) = \frac{s_l(1-s_l)\theta}{e - s_l \theta},$$

which satisfies differential equation (2). Matching coefficients, we have $r = s_l(1-s_l) > 0$, $\theta_0 = 0 \in \Theta$. Therefore, by Theorem 2, the unique Riley equilibrium is linear, which coincides with Lemma 3 and Proposition 5 of Hermalin (1998).[12] Yet this result hinges critically on the assumption that $\Theta = [0,1]$. If the productivity factor is bounded away from 0 (for example, $\Theta = [1,2]$), then $\theta_0 = 0 \notin \Theta$. By Theorem 2, Riley equilibrium is no longer linear. This case is not analyzed by Hermalin (1998).

To deepen our understanding, we rewrite the leader's problem as follows:

$$V(\theta, \hat{\theta}, e) = -\frac{1}{2}(e - s_l \theta)^2 + \frac{1}{2}s_l^2 \theta^2 + r \theta \hat{\theta}.$$

If the leader is myopic and ignores the inferential impact of his effort, he optimally chooses $e = s_l \theta$—that is, his bliss point. A strategic leader can benefit from inducing a higher belief $\hat{\theta}$. The marginal benefit of inducing a higher belief is $\frac{\partial V}{\partial \hat{\theta}} = r\theta$, where r is the marginal impact. Thus, the higher the productivity factor θ, the stronger the incentive. As the leader desires to induce a higher belief, $e = s_l \theta$ cannot be sustained in equilibrium and his effort is biased upward. But as his effort diverges from his

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[12]Zhou (2016) uses Hermalin's (1998) model to explore leadership within hierarchical organizations. His analysis heavily depends on the linear equilibrium.
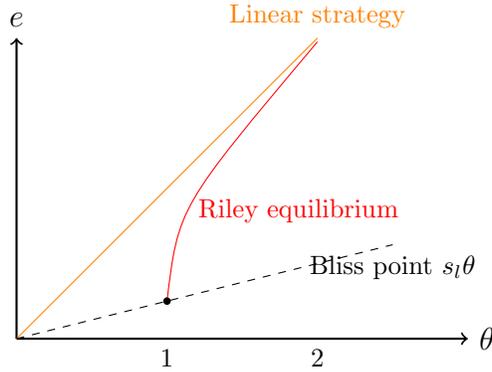
Figure 4: The Leader's Strategy for $\Theta = [1, 2]$

bliss point $e = s_l\theta$, it incurs a cost in $-\frac{1}{2}(e - s_l\theta)^2$. In equilibrium, for all separating strategies, the marginal cost of exerting more effort must equal the marginal benefit of inducing a higher belief. Among these strategies, the strategy in Riley equilibrium is closest to the leader's bliss point $e = s_l\theta$.

Suppose that $\Theta = [1, 2]$. Riley equilibrium is between the leader's bliss point and the incentive-compatible linear strategy (Figure 4) and therefore features a uniformly lower effort than the incentive-compatible linear strategy.[13] In the linear strategy, the marginal cost of inducing a higher belief

$$\left.\frac{\partial}{\partial \hat{\theta}}[-\frac{1}{2}(e(\hat{\theta}) - s_l\theta)^2]\right|_{\hat{\theta}=\theta}$$

is a first-order effect, as $e(\theta) - s_l\theta > 0$ for all $\theta \in \Theta$. The linear slope is pinned down such that this first-order effect counterbalances the marginal benefit $r\theta$. By contrast, Riley equilibrium features $e(1) = s_l$. If $e(\theta)$ were to increase linearly, the marginal cost of inducing a higher belief

$$\left.\frac{\partial}{\partial \hat{\theta}}[-\frac{1}{2}(e(\hat{\theta}) - s_l\theta)^2]\right|_{\hat{\theta}=\theta}$$

would be zero at $\theta = 1$ (a second-order effect), which falls short of the nonzero marginal benefit. Thus, as θ approaches 1, the slope of $e(\theta)$ tends to infinity. As θ increases, $e(\theta)$ diverges away from $s_l\theta$. As the strategic distortion $e(\theta) - s_l\theta$ increases, it requires less of an increment in $e(\theta) - s_l\theta$ to counterbalance the marginal benefit.

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[13]See the closed form in the proof of Theorem 2 in Appendix C.

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But as the strategic distortion in Riley equilibrium is smaller than that of the linear strategy, the slope $e'(\theta)$ of Riley equilibrium is larger than that of the linear strategy. Therefore, Riley equilibrium strategy converges to the linear strategy as θ increases. Consequently, Riley equilibrium is nonlinear.

What is different when $\Theta = [0,1]$? If $\Theta = [0,1]$, then at $\theta = 0$, the marginal benefit of inducing a higher belief, $r\theta$, is also zero, which allows the linear strategy with $e(0) = 0$ to grow linearly at $\theta = 0$. As the linear strategy satisfies the initial condition $e(0) = 0$, the linear strategy coincides with Riley equilibrium.

5.2   Data Linkages

Argenziano and Bonatti (2021) consider a dynamic model of behavior-based price discrimination. (They use a different solution concept—Bayesian Nash equilibrium.[14]) A consumer sequentially interacts with two firms. In each period $t \in \{1,2\}$, the active firm sets a price p_t and a quality level y_t and the consumer chooses the quantity q_t to consume. A data linkage allows the second firm to observe the first-period interaction outcome (p_1, y_1, q_1). After observing the outcome, the second firm tailors its quality level and price to the consumer's type.

In each period, the consumer's utility is given by

$$U(p_t, y_t, q_t) = (\theta + b_t y_t - p_t)q_t - \frac{q_t^2}{2},$$

where the consumer's type $\theta \in \Theta = [m, M]$ is his baseline consumption level before adjusting for price and quality, $b_t \in [0, \sqrt{2})$ is common knowledge and represents the sensitivity of the consumer's valuation to the quality of firm t's product, and $b_t y_t - p_t$ is the terms of trade that firm t offers to the consumer. Firm t's profits are

$$\Pi(p_t, y_t, q_t) = p_t q_t - \frac{y_t^2}{2}.$$

To maximize profits, the second firm sets its terms of trade to be $\lambda_2 \hat{\theta}$ (we skip the calculations), where $\hat{\theta}$ is the inferred type and $\lambda_2 = \frac{b_2^2 - 1}{2 - b_2^2} \in [-\frac{1}{2}, \infty)$. Given first-period

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[14]They require the firm's strategy to be linear even off the path as in Ball (2021). See footnote 6 in Argenziano and Bonatti (2021) for more details.

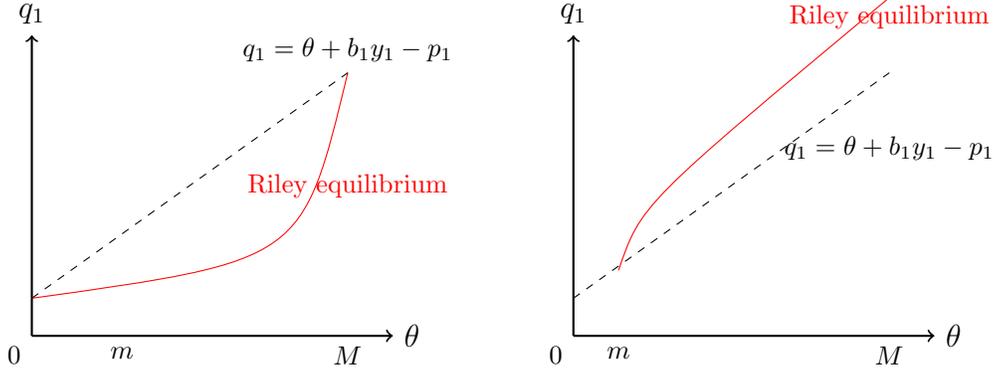Figure 5: Consumer's Strategy for $\lambda_2 < 0$ (Left), $\lambda_2 > 0$ and $0 \notin \Theta$ (Right)

consumption q_1 and inference $\hat{\theta}$, the type-θ consumer's payoff is

$$V(\theta, \hat{\theta}, q_1) = (\theta + b_1 y_1 - p_1)q_1 - \frac{q_1^2}{2} + \frac{1}{2}(\theta + \lambda_2\hat{\theta})^2.$$

Therefore, the consumer solves the problem

$$\max_{q_1(\hat{\theta})} [\underbrace{(\theta + b_1 y_1 - p_1)q_1(\hat{\theta}) - \frac{q_1^2(\hat{\theta})}{2}}_{\text{first-period utility}} + \underbrace{\frac{1}{2}(\theta + \lambda_2\hat{\theta})^2}_{\text{second-period utility}}]. \qquad (4)$$

Matching the coefficients to our model (1), we have $r = \lambda_2(1 + \lambda_2)$, $\theta_0 = 0$. Let $S = q_1 - (\theta + b_1 y_1 - p_1)$. There are two linear IC strategies $S = \lambda_2\theta$ and $S = -(\lambda_2 + 1)\theta$. By Theorem 2, in Riley equilibrium, the consumer's strategy is linear iff $\lambda_2 > 0$ and $0 \in \Theta$. When Riley equilibrium is indeed linear, it coincides with the linear BNE of Proposition 2 in Argenziano and Bonatti (2021).

Yet our Theorem 2 implies that Riley equilibrium can be nonlinear.[15] We plot the remaining cases in Figure 5. In the first period, the consumer strategically consumes

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[15]When $\lambda_2 < 0$, Riley equilibrium strategy is

$$\frac{(S - \lambda_2\theta)^{\lambda_2}}{[S + (\lambda_2 + 1)\theta]^{-(\lambda_2+1)}} = \frac{(-\lambda_2 M)^{\lambda_2}}{(\lambda_2 M + M)^{-(\lambda_2+1)}} \qquad (S \leq 0).$$

When $\lambda_2 > 0$ and $0 \notin \Theta$, Riley equilibrium is

$$\frac{(\lambda_2\theta - S)^{\lambda_2}}{[S + (1 + \lambda_2)\theta]^{-(1+\lambda_2)}} = \frac{(\lambda_2 m)^{\lambda_2}}{[(1 + \lambda_2)m]^{-(1+\lambda_2)}} \qquad (S \geq 0).$$

We need to additionally check whether the strategy in Theorem 2 satisfies problem (4)'s second-order condition. It turns out to be true for both of the above solutions.

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less than his ideal quantity $\theta + b_1 y_1 - p_1$ if $\lambda_2 < 0$ but consumes more than his ideal quantity if $\lambda_2 > 0$. When $\lambda_2 < 0$, the second firm's terms of trade $\lambda_2 \hat{\theta}$ are decreasing in belief. As the consumer's second-period utility is $\frac{1}{2}(\theta + \lambda_2 \hat{\theta})^2$, the consumer bears a cost of inducing a higher belief. In equilibrium, as the consumer consumes less than the ideal quantity, the marginal benefit of consuming more in the first period must counterbalance the marginal cost of inducing a higher belief in the second period. By contrast, when $\lambda_2 > 0$, the consumer benefits from inducing a higher belief. In equilibrium, as the consumer consumes more than the ideal quantity, the marginal cost of consuming more in the first period must counterbalance the marginal benefit of inducing a higher belief in the second period.

5.3 Strategic Communication with Lying Costs

Kartik (2009) and Kartik et al. (2007) analyze a model of strategic communication between an informed but upwardly biased sender (he) and an uninformed receiver (she). The sender bears a cost of misreporting or lying about his private information. The cost may stem from moral constraints, legal penalties, or fabrication costs. In this setting, the sender may employ inflated language, where by *inflated* we mean that the sender's message is biased above the state. To preserve the flavor of inflated language in equilibrium without getting into technical details, we assume that $\Theta = [0, 1]$ as in Kartik (2009).

In the model, a sender is privately informed about the state $\theta \in \Theta$. After observing the state, he sends a message $\mathfrak{m}$ to a receiver, who then takes an action a_2. The sender's payoff is

$$U^S = -k(\mathfrak{m} - \theta)^2 - (a_2 - a_2^S(\theta))^2,$$

where $a_2^S(\theta) = \lambda \theta + b$ is his ideal action for the receiver and $k(\mathfrak{m} - \theta)^2$ denotes the cost of lying when the state is θ and the message is $\mathfrak{m}$. The receiver's payoff is maximized when her action a_2 matches the state—that is, $a^R(\theta) = \theta$. Given the receiver's best response, the sender gets a payoff of

$$V(\theta, \hat{\theta}, \mathfrak{m}) = -k(\mathfrak{m} - \theta)^2 - [\hat{\theta} - (\lambda \theta + b)]^2$$

when the receiver's belief is $\hat{\theta}$. Plugging in the coefficients from our model (1), we get $r = \frac{\lambda - 1}{k}$ and $\theta_0 = -\frac{b}{\lambda - 1}$.
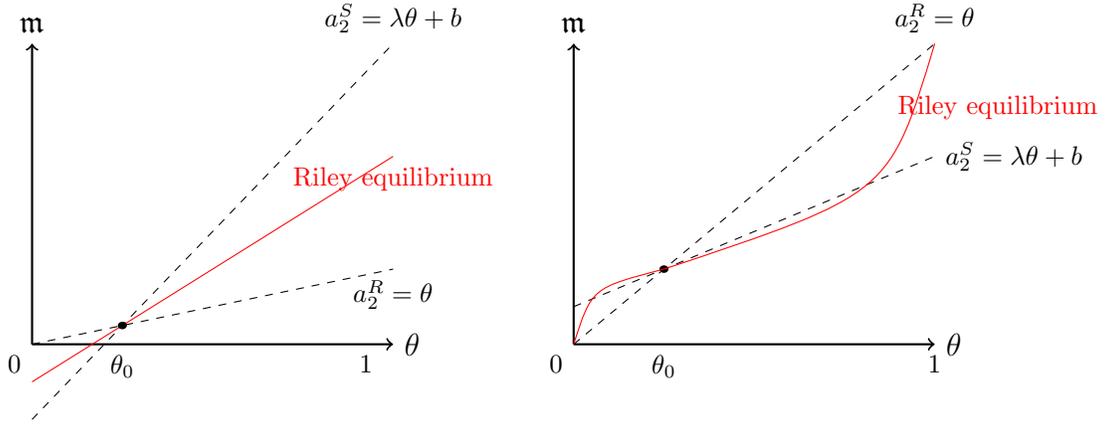
Figure 6: Sender's Strategy when $\theta_0 \in \Theta$
$\lambda > 1$ (Left) and $\lambda < 1$ (Right)

Kartik et al. (2007) consider the case where $\lambda = 1$ and $b > 0$. By Theorem 2, the sender's strategy is

$$\mathfrak{m} + \frac{b}{k} \ln(\frac{b}{k} - \mathfrak{m} + \theta) = \frac{b}{k} \ln(\frac{b}{k}),$$

which is nonlinear. It coincides with solution (6) in Kartik et al. (2007). The language in equilibrium is inflated, as $\mathfrak{m} \geq \theta$.

Yet this feature of inflated language relies critically on local belief monotonicity—that is, $a_2^S(\theta) > a^R(\theta)$ for all $\theta \in \Theta$. (See Section 6 for a formal definition of belief monotonicity.) What happens if the sender's most-preferred action $a_2^S(\theta)$ is biased upward for some states but biased downward for other states? Then, the sender may employ either inflated or deflated language and the direction of language distortion depends only on how the sender's bliss point $a_2^S(\theta)$ is biased from the receiver's ideal point $a^R(\theta)$. In particular, the language is inflated (deflated) whenever the sender's bliss point is above (below) the state.

Proposition 3. *In Riley equilibrium, the sender's language is inflated at θ if and only if $a_2^S(\theta) > a^R(\theta)$.*

This result is not limited to the model of lying cost. A similar conclusion holds for all quadratic signaling games. By Theorem 2, Riley equilibrium is linear if and only if $\lambda > 1$ and $\theta_0 \in \Theta$. In Figure 6, we plot the case for $\lambda > 1$ when Riley equilibrium is linear and the case for $\lambda < 1$ when Riley equilibrium is nonlinear. We can see that the direction of the language distortion is indeed aligned with the direction of the preference bias $a_2^S(\theta) - a^R(\theta)$.
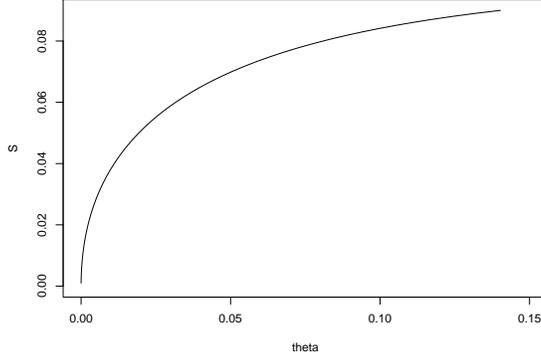
Figure 7: Dominant Delegation
This figure is drawn for $k_1 = k_2 = 1$, $b_1 = b_2 = 0.2$, $\delta = 0.5$.

5.4 Transferable Control and Learning by Delegation

The classic delegation problem considers a principal (she) who faces an informed but biased agent (he). The principal delegates control to the agent to use his local knowledge and is unable to commit to contingent transfers. In many real-life examples, the principal cannot contractually commit to relinquishing control in the future (Aghion et al., 2004). After learning the local knowledge from the agent's decision, the principal can reclaim control and make decisions by herself. Aghion et al. (2004) study transferable control and learning by delegation when the state is discrete. In this section, we study the same problem when the state is continuous.

In a two-period delegation model, the principal delegates control in the first period and reclaims control in the second one. Let $\theta \in \Theta = [0, 1]$ denote the state of the world. In the first period, the agent privately observes the state θ and makes a decision $a_1 \in \mathbb{R}$. In the second period, the principal makes a decision $a_2 \in \mathbb{R}$ based on the agent's decision a_1. The principal's and agent's payoffs depend on the implemented decisions and the state. The principal wants her decision a_2 to match the state, while the agent's ideal point is $a_1^S = a_2^S = \theta + b$, where $b > 0$ measures the preference bias. The agent's payoff is $U^S = -(a_1 - \theta - b)^2 - \delta(a_2 - \theta - b)^2$.

Given the principal's belief $\hat{\theta}$, the agent's payoff is

$$V(\theta, \hat{\theta}, a_1) = -(a_1 - \theta - b)^2 - \delta(\hat{\theta} - \theta - b)^2.$$

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Matching the coefficients to model (1), we have $r = 0$. By Theorem 2, Riley equilibrium is nonlinear (see Figure 7):

$$\theta + S + \delta b \ln(\delta b - S) = \delta b \ln(\delta b).$$

As δ tends to zero, Riley equilibrium degenerates to the standard static delegation

$$a_1 = \theta + b$$

as in Dessein (2002).

Thus far, we have focused on Riley equilibrium that maximizes the sender's (i.e., the agent's) payoff at every state. In this example, we explore how the receiver (i.e., the principal) ranks different separating PBE. Does Riley equilibrium yield the highest possible expected payoff to the principal? To address this question, it is necessary to impose some assumptions on the state distribution and on the principal's preference about a_1. As an extension of the classic example in Crawford and Sobel (1982), we assume that the state is uniformly distributed and that $U^R = -(a_1 - \theta)^2 + g(\theta, a_2)$, where $g(\theta, a_2)$ is maximized at $a_2 = \theta$ for all θ. In this setting, we can show that Riley equilibrium is also optimal for the principal among all separating equilibria.

Proposition 4. *If the state is uniformly distributed, Riley equilibrium is optimal for the principal.*

By the proof of Theorem 2, we can analytically solve for all IC strategies. Some of them are discontinuous. To prove Proposition 4, we first show that all discontinuous strategies are Pareto-dominated by the linear strategy. (The linear strategy is increasing.) Second, we show that all decreasing continuous strategies are inadmissible for any PBE. Third, among all increasing continuous strategies, we show that Riley equilibrium is uniformly closest to the principal's ideal point.

6   Concluding Remarks

This paper studied Riley equilibria in quadratic signaling games. We derived a necessary and sufficient condition for the existence and uniqueness of a linear Riley equilibrium. One assumption we imposed is a quadratic form of the preference. This form

succinctly captures some key properties of concave preferences and allows for closed-form solutions. Nevertheless, a natural direction for future research is to generalize our conclusions beyond quadratic games.

In this section, we conclude our paper by showing why quadratic signaling games do not necessarily satisfy some widely adopted assumptions in the literature, including the belief monotonicity and single-crossing conditions. Then we explore some properties of Riley equilibrium in quadratic signaling games.

6.1 Belief Monotonicity

In the literature (see, for example, Mailath, 1987; Roddie, 2011; Kartik et al., 2007), the belief-monotonicity assumption states that

$$V_2(\theta, \hat{\theta}, a_1) \neq 0 \qquad \forall (\theta, \hat{\theta}, a_1),$$

where subscripts on functions denote derivatives. That is, the sender always prefers a higher (lower) belief, regardless of the state, the belief, and his action. Belief monotonicity is satisfied if and only if $a_2^S(\theta) \notin \Theta$ for all $\theta \in \Theta$. To see this, if $a_2^S(\theta) \in \Theta$ for some $\theta \in \Theta$, then $V_2(\theta, a_2^S(\theta), a_1) = 0$ for all a_1. In the delegation example, as long as b is sufficiently small, there exists some $\theta \in \Theta$ such that $a_2^S(\theta) \in \Theta$.

Kartik et al. (2007) consider a weaker condition

$$V_2(\theta, \theta, a_1) \neq 0 \qquad \forall (\theta, a_1),$$

which we call *local belief monotonicity*. That is, given a correct belief, the sender always prefers a slightly higher (lower) belief, regardless of the state and his action. Local belief monotonicity is satisfied if and only if $\theta_0 \notin \Theta$, as $V_2(\theta, \theta, a_1) = 0$ if and only if $\theta = \theta_0$.

6.2 Single-Crossing Condition

We show that our model does not necessarily satisfy the single-crossing condition. A function $V(\theta, \hat{\theta}, a_1)$ satisfies *single-crossing* if when $\theta_1 < \theta_2$ and $\hat{\theta}_1 < \hat{\theta}_2$: $V(\theta_1, \hat{\theta}_1, a_1) \leq V(\theta_1, \hat{\theta}_2, a_1')$ and $a_1 \leq a_1'$ imply $V(\theta_2, \hat{\theta}_1, a_1) \leq V(\theta_2, \hat{\theta}_2, a_1')$, and strictness in either inequality implies $V(\theta_2, \hat{\theta}_1, a_1) < V(\theta_2, \hat{\theta}_2, a_1')$. A function $V(\theta, \hat{\theta}, a_1)$ satisfies *strong*
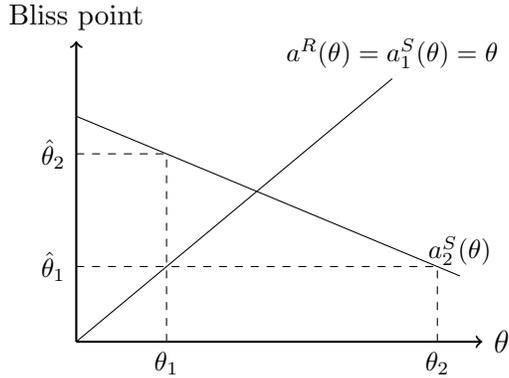
Figure 8: Players' Bliss Points

single-crossing if, when $\theta_1 < \theta_2$, $V(\theta_1, \hat{\theta}_1, a_1) \leq V(\theta_1, \hat{\theta}_2, a_1')$ and $a_1 \leq a_1'$ imply $V(\theta_2, \hat{\theta}_1, a_1) \leq V(\theta_2, \hat{\theta}_2, a_1')$, and strictness in either inequality implies $V(\theta_2, \hat{\theta}_1, a_1) < V(\theta_2, \hat{\theta}_2, a_1')$. Strong single-crossing implies single-crossing. Moreover, the Spence–Mirrlees single-crossing condition implies strong single-crossing (Cho and Sobel, 1990).

Consider a setting in which $a^R(\theta) = a_1^S(\theta) = \theta$ and $a_2^S(\theta)$ has a negative slope. We set $\hat{\theta}_1 = \theta_1 < \theta_0$, θ_2 such that $a_2^S(\theta_2) = \hat{\theta}_1$, and $\hat{\theta}_2$ such that $\hat{\theta}_2 = a_2^S(\theta_1)$ (see Figure 8). Obviously $\theta_1 < \theta_2$ and $\hat{\theta}_1 < \hat{\theta}_2$. Next we set $a_1' = a_1^S(\theta_1) + d$, $a_1 = a_1^S(\theta_1) - d$, for some $d > 0$. By construction, we have $a_1' > a_1$ and $V(\theta_1, \hat{\theta}_1, a_1) < V(\theta_1, \hat{\theta}_2, a_1')$, which also holds if we replace θ_1 by $\tilde{\theta}_2 > \theta_1$ sufficiently close to θ_1. But

$$V(\theta_2, \hat{\theta}_2, a_1') - V(\theta_2, \hat{\theta}_1, a_1) = 4d(\theta_2 - \theta_1) - \delta(\hat{\theta}_2 - \theta_1)^2.$$

Since $d > 0$ can be made arbitrarily small, we have $V(\theta_2, \hat{\theta}_1, a_1) > V(\theta_2, \hat{\theta}_2, a_1')$ for some sufficiently small d.

6.3 Comparative Static Analysis

We now perform a comparative static analysis. To begin, we illustrate the sender's manipulation incentives. In any separating equilibrium, the sender's action a_1 signals his type. We let $\mu(a_1)$ express the belief's dependence on the sender's action. He solves the following problem:

$$\max_{a_1}[-(a_1 - k_1\theta - b_1)^2 - \delta(\mu(a_1) - k_2\theta - b_2)^2]$$

The sender thereby faces a trade-off between optimizing his action a_1 and manipulating $\hat{\theta}$. The strength of such manipulation incentives hinges on δ and the sensitivity of the belief $\mu(a_1)$. The greater δ, the stronger the manipulation incentive, which leads to a larger strategic distortion.

Proposition 5. *Suppose that the discriminant is non-negative. In Riley equilibrium, $|S|$ is increasing in δ.*

6.4 Properties of Riley Equilibrium

In this subsection, we highlight some properties of Riley equilibrium. First, given belief monotonicity, we illustrate how Riley equilibrium is consistent with the classic initial condition of Mailath (1987). This condition requires that $S(\theta^w) = 0$, where θ^w is the worst type—that is, the worst point belief the receiver can hold. For instance, if $V_2(\theta, \hat{\theta}, a_1) > 0$ for all $(\theta, \hat{\theta}, a_1)$, then $\theta^w = m$. In Mailath (1987), this condition and incentive compatibility together can uniquely pin down an equilibrium. We show that this initial condition is implied by Riley equilibrium, given belief monotonicity. Second, we generalize this result as we gradually relax belief monotonicity.

The initial condition is justified by sequentiality. That is, since θ^w is the worst belief, if $S(\theta^w) \neq 0$, a deviation by $S(\theta^w) \neq 0$ to $S(\theta^w) = 0$ cannot be credibly punished in equilibrium. However, sequentiality is a pure equilibrium-refinement condition and hence does not explain why the sender does so in the first place.

To connect our discussion to Mailath (1987), consider a setting satisfying belief monotonicity. Without loss of generality, we take $a_2^S(\theta) > M$ for all θ as an illustration. In this setting, $V_2(\theta, \hat{\theta}, a_1) > 0$ for all $(\theta, \hat{\theta}, a_1)$. Thus, the worst type $\theta^w = m$. Then Riley equilibrium features $S(m) = 0$. That is, the traditional initial condition is implied by the optimality of the sender's strategy. That is, the initial condition comes as part of Riley equilibrium.

To generalize this result, let us modify the above example. Without loss of generality, consider $k_2 > 1$ and $a_2^S(m) \in \Theta$. Then the belief monotonicity condition is violated. Riley equilibrium in this setting still features $S(m) = 0$. But how do we extrapolate from the above intuition? The answer lies in local belief monotonicity. In this example, since $V_2(\theta, \theta, a_1) > 0$ for all (θ, a_1), we can similarly define a *generalized worst type* $\theta^w = m$, and Riley equilibrium still has $S(\theta^w) = 0$.

What happens if the local belief monotonicity is violated? This occurs if and

only if $\theta_0 \in \Theta$. In this situation, Riley equilibrium features $S(\theta_0) = 0$. Yet it is inappropriate to view this as an initial condition since θ_0 is a singularity of the differential equation. At $\theta = \theta_0$, in the Riley equilibrium, the sender obtains the maximum payoff possible, $U^S = 0$. That is, the sender achieves his ideal points for both a_1 and a_2. Since preferences are aligned at θ_0, the sender willingly reveals his type. Then the receiver takes the action most beneficial for both of them.

We can summarize the above discussion in the following proposition. Let θ^w denote the generalized worst type when local belief monotonicity holds. Formally, $\theta^w = m$ if $V_2(\theta, \theta, a_1) > 0$ for all (θ, a_1), and $\theta^w = M$ if $V_2(\theta, \theta, a_1) < 0$ for all (θ, a_1).

Proposition 6. *Suppose that the discriminant is non-negative. In Riley equilibrium, $S(\theta^w) = 0$ if local belief monotonicity holds, and $S(\theta_0) = 0$ otherwise.*

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Appendix

A Solutions to Differential Equation (2)

We start by solving differential equation (2). The solutions will be useful later. We solve it case by case.

Analysis for $r = 0$

The DE reduces to

$$S(\theta)\frac{dS(\theta)}{d\theta} = (\delta b_2 - k_1 S(\theta)).$$

The solution is as follows:

$$k_1^2 \theta + k_1 S(\theta) + \delta b_2 \ln |\delta b_2 - k_1 S(\theta)| = C.$$

A special solution is $k_1 S(\theta) = \delta b_2$.

Analysis for $r \neq 0$

Define $\phi \triangleq \theta - \theta_0$. DE (2) is

$$S(\phi)[\frac{dS(\phi)}{d\phi} + k_1] = r\phi. \tag{5}$$

It is centrosymmetric around $(\phi, S) = (0, 0)$. Let $w(\phi) \triangleq \frac{S(\phi)}{\phi}$. DE (5) reduces to

$$\frac{w(\phi)}{w^2(\phi) + k_1 w(\phi) - r}\frac{dw(\phi)}{d\phi} = -\frac{1}{\phi}. \tag{6}$$

The form of general solutions depends on Δ. A linear solution exists if and only if $\Delta \geq 0$. If $\Delta > 0$, there exist two linear special solutions, $S = w_1 \phi$ and $S = w_2 \phi$, where w_1 and w_2 denote solutions to the equation $w^2 + k_1 w - r = 0$. Without loss of generality, let us assume $|w_1| \leq |w_2|$.

$$w_1 + w_2 = -k_1 \tag{7}$$

$$w_1 \times w_2 = -r \tag{8}$$

We further rewrite DE (6):

$$\frac{w(\phi)dw(\phi)}{(w(\phi) - w_1)(w(\phi) - w_2)} = -\frac{d\phi}{\phi}$$

$$\frac{w_1}{w_1 - w_2}\frac{dw(\phi)}{w(\phi) - w_1} - \frac{w_2}{w_1 - w_2}\frac{dw(\phi)}{w(\phi) - w_2} = -\frac{d\phi}{\phi}$$

The general solutions take the form

$$\frac{|S(\phi) - w_1\phi|^{w_1}}{|S(\phi) - w_2\phi|^{w_2}} = C. \tag{9}$$

If $\Delta = 0$, there is a unique linear special solution $S(\phi) = w_1\phi$, where $w_1 = w_2 = -\frac{k_1}{2}$ is the solution for equation $w^2 + k_1 w - r = 0$. We further rewrite DE (6) as

$$\frac{w(\phi)dw(\phi)}{(w - w_1)^2} = -\frac{d\phi}{\phi}$$

$$\frac{dw(\phi)}{w(\phi) - w_1} + \frac{w_1 dw(\phi)}{(w(\phi) - w_1)^2} = -\frac{d\phi}{\phi}.$$

The general solutions take the form

$$\ln|S(\phi) - w_1\phi| - \frac{w_1\phi}{S(\phi) - w_1\phi} = C. \tag{10}$$

If $\Delta < 0$, define $q \triangleq \frac{1}{2}\sqrt{-\Delta}$, $z(\phi) \triangleq w(\phi) + \frac{k_1}{2}$. We rewrite DE (6) as follows:

$$\frac{w(\phi)dw(\phi)}{(w(\phi) + \frac{k_1}{2})^2 - \frac{\Delta}{4}} = -\frac{d\phi}{\phi}$$

$$\frac{zdz}{z^2 + q^2} - \frac{k_1}{2}\frac{dz}{z^2 + q^2} = -\frac{d\phi}{\phi}$$

The general solutions are

$$\ln|S^2 + k_1 S\phi - r\phi^2| - \frac{k_1}{q}\arg\tan(\frac{S}{q\phi} + \frac{k_1}{2q}) = C. \tag{11}$$

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B   Omitted Proofs in Section 3

We first prove four Lemmas. Then the proof of Proposition 1 follows.

Lemma 1. *Continuity*. *Suppose a separating strategy S is incentive compatible. Then $|S|$ is continuous.*

Proof of Lemma 1. If the function $|S|$ is discontinuous at $\tilde{\theta}$, the sender's equilibrium payoff $-S^2(\theta) - \delta(\theta - a_2^S(\theta))^2$ is also discontinuous at $\tilde{\theta}$. Then incentive compatibility fails in an arbitrarily small neighborhood of $\tilde{\theta}$. Thus, the function $|S|$ must be continuous everywhere. □

For the next three lemmas, we only prove the case where $k_1 > 0$. The other case is symmetric around the θ-axis.

Lemma 2. *Aligned Monotonicity*.

1. *Suppose $k_1 > 0$. For any $\theta \in \Theta$, if there exists a sequence $\theta_n \to \theta$ and $\theta_n \leq \theta$ for all n such that $S(\theta_n) > 0$, then $S(\theta) \geq 0$.*

2. *Suppose $k_1 > 0$. For any $\theta \in \Theta$, if there exists a sequence $\theta_n \to \theta$ and $\theta_n \geq \theta$ for all n such that $S(\theta_n) < 0$, then $S(\theta) \leq 0$.*

Proof of Lemma 2. Since the proofs of these two claims are similar, we just prove the first one.

By way of contradiction, suppose $S(\theta) < 0$. Take θ_n, θ_m from the sequence. Let $\theta_n \neq \theta_m$ tend to θ, by $|S|$ being continuous, $S(\theta_n) \to |S(\theta)|$, $S(\theta_m) \to |S(\theta)|$. We first prove

$$\lim_{n,m \to \infty} \frac{S(\theta_n) - S(\theta_m)}{\theta_n - \theta_m} = \frac{r\theta + \delta b_2}{|S(\theta)|} - k_1.$$

Let $T(\theta) = \frac{r\theta + \delta b_2}{|S(\theta)|} - k_1$. Suppose there exists an $\epsilon > 0$ such that for any $N > 0$, there exists $n, m > N$ with $\theta_n \neq \theta_m$ such that

$$\left| \frac{S(\theta_n) - S(\theta_m)}{\theta_n - \theta_m} - T(\theta) \right| > \epsilon.$$

If

$$\frac{S(\theta_n) - S(\theta_m)}{\theta_n - \theta_m} > T(\theta) + \epsilon,$$

$$\mathscr{U}(\theta_m; \theta_n) - \mathscr{U}(\theta_n; \theta_n) > 2\epsilon|S(\theta)(\theta_n - \theta_m)| + \mathcal{O}((\theta_n - \theta_m)^2);$$

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if

$$\frac{S(\theta_n) - S(\theta_m)}{\theta_n - \theta_m} < T(\theta) - \epsilon,$$

$$\mathscr{U}(\theta_n; \theta_m) - \mathscr{U}(\theta_m; \theta_m) > 2\epsilon|S(\theta)(\theta_n - \theta_m)| + \mathcal{O}((\theta_n - \theta_m)^2),$$

which contradicts incentive compatibility when $|\theta_n - \theta_m|$ is sufficiently small.

Then fix an n and let $m \to \infty$. We have

$$S(\theta_n) = |S(\theta)| - (\frac{\delta b_2 + r\theta}{|S(\theta)|} - k_1)(\theta - \theta_n) + \mathcal{O}((\theta - \theta_n)^2). \tag{12}$$

By the definition of $\mathscr{U}$,

$$\mathscr{U}(\theta; \theta_n) = -[|S(\theta)| - k_1(\theta - \theta_n)]^2 - \delta(\theta - k_2\theta_n - b_2)^2,$$

$$\mathscr{U}(\theta_n; \theta_n) = -S^2(\theta_n) - \delta(\theta_n - k_2\theta_n - b_2)^2.$$

By (12), the difference

$$\mathscr{U}(\theta; \theta_n) - \mathscr{U}(\theta_n; \theta_n) = 4k_1|S(\theta)|(\theta - \theta_n) - \mathcal{O}((\theta - \theta_n)^2) > 0$$

when $\theta - \theta_n$ is sufficiently small. This implies that if $S(\theta) < 0$, the sender at sufficiently close θ_n has incentive to mimic θ; hence the incentive-compatibility constraint fails. So it must be true that $S(\theta) \geq 0$. $\qquad \square$

Lemma 3. *Jump*. *Suppose a separating strategy is incentive compatible. If it is discontinuous, at any discontinuity, both left and right limits exist and the jump direction has the same sign as k_1.*

Proof of Lemma 3. As $|S|$ is continuous, $|S(\theta-)| = |S(\theta)| = |S(\theta+)|$. Suppose $S(\theta)$ is discontinuous at y. Then $|S(y)| \neq 0$. It suffices to prove the case of $S(y) > 0$. The other case is symmetric. Suppose $S(y) > 0$. By the contrapositive of the second point of Lemma 2, there exists an $\epsilon > 0$ such that $S(\theta) \geq 0$ for all $\theta \in (y, y + \epsilon)$. Since $|S|$ is continuous, the right limit $S(y+)$ exists and $S(y) = S(y+) > 0$. Then either $S(y-)$ does not exist or $S(y-)$ exists and $S(y-) = -S(y+)$. Next we show that $S(y-)$ exists.

By way of contradiction, suppose $S(y-)$ does not exist. As $|S|$ is continuous, there exists a sequence $y_n \to y$ and $y_n < y$ such that $S(y_n) > 0$. Take a y_m sufficiently close

to y such that $|S(\theta)| > 0$ for all $\theta \in [y_m, y]$. Define

$$x \triangleq \sup\{\theta' \in \Theta | S \text{ is continuous on } (y_m, \theta')\}.$$

By the contrapositive of the second point of Lemma 2 and $S(y_m) > 0$, $x > y_m$. And since $S(y-)$ does not exist, $x < y$. If $S(x) > 0$, by the contrapositive of the second point of Lemma 2, there exists an $\epsilon' > 0$ such that $S(\theta) \geq 0$ for all $\theta \in (x, x + \epsilon')$. This implies S is continuous on $\theta \in (y_m, x + \epsilon')$, contradicting the definition of x. If $S(x) < 0$, by the contrapositive of the first point of Lemma 2, there exists an $\epsilon'' > 0$ such that $S(\theta) \leq 0$ for all $\theta \in (x - \epsilon'', x)$, contradicting the definition of x. $\qquad\square$

Lemma 4. *Differentiability*. *Suppose a separating strategy S is incentive compatible. Then at every point in (m, M) where S is continuous, S is differentiable and satisfies* (2) *and* (3).

Proof of Lemma 4. The proof of Theorem 1 in Mailath (1987) establishes the differentiability of S when $S(\theta) \neq 0$.[16] Next we analyze the case when S is zero. Suppose S is continuous at some $t \in (m, M)$ and $S(t) = 0$.

First, we rule out the case in which $k_2 t + b_2 \neq t$ ($t \neq \theta_0$). Suppose $k_2 t + b_2 < t$. (The case in which $k_2 t + b_2 > t$ is proven symmetrically.) Take an $\epsilon > 0$ sufficiently small such that $k_2(t + \epsilon) + b_2 < t + \epsilon$. Then the sender at state $t + \epsilon$ has a strict incentive to mimic state t since

$$\mathscr{U}(t; t + \epsilon) - \mathscr{U}(t + \epsilon; t + \epsilon) = S^2(t + \epsilon) + 2\delta(t - k_2 t - b_2)\epsilon + \mathcal{O}(\epsilon^2),$$

which is strictly positive as ϵ is small, contradicting incentive compatibility.

For the remaining proof, we assume $k_2 t + b_2 = t$. Suppose $r = 0$; then $k_2 = 1$ and $b_2 = 0$. Then by DE (2), the sender's strategy is $S = 0$, which is differentiable. Suppose $r \neq 0$. As $k_2 \neq 1$, it must be true that $t = \theta_0$. For any $\theta \neq \theta_0$, by the first part, $S(\theta) \neq 0$. By the Jump Lemma, there exists an ϵ' such that S is continuous on $(t - \epsilon', t + \epsilon')$. As S is nonzero and continuous on $(t - \epsilon', t)$, the behavior of S is governed by differential equation (2). Now we analyze this case by case in terms of Δ.

---

[16]On page 1361 of Mailath (1987), the proof of Theorem 1 consists of four propositions. Proposition 2 proves differentiability. To adopt his proof, we only need to redefine his Y to be $Y \triangleq \{y \in \mathscr{R} : \exists\theta, V(\theta, \theta, y) \geq V(\theta, m, \sigma(m))\}$.
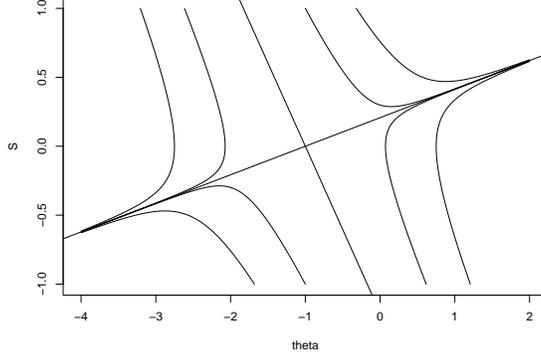
Figure 9: Solutions for $r > 0$
This figure is drawn for $k_1 = 1$, $k_2 = 1.5$, $b_1 = b_2 = \delta = 0.5$.

It is impossible that $\Delta < 0$ since no separating solution (11) crosses $(\theta, S) = (\theta_0, 0)$.[17]
Now suppose $\Delta \geq 0$. If $r > 0$, only two linear solutions cross $(\theta, S) = (\theta_0, 0)$ (Figure 9), and they are differentiable. Supposing $r < 0$, all solutions cross $(\theta, S) = (\theta_0, 0)$. As linear solutions are always differentiable, we only need to show that nonlinear solutions are differentiable at θ_0. We prove the stronger result that the derivative of all nonlinear solutions is w_1.

Let us analyze the right derivative of all nonlinear solutions at θ_0. Take a sufficiently small ϵ''. It must be true that $\frac{S(\theta)}{\theta - \theta_0} \in (w_1, 0)$ for all $\theta \in (\theta_0, \theta_0 + \epsilon'')$ or $\frac{S(\theta)}{\theta - \theta_0} \in (w_2, w_1)$ for all $\theta \in (\theta_0, \theta_0 + \epsilon'')$ (Figure 10). It suffices to consider $\frac{S(\theta)}{\theta - \theta_0} \in (w_1, 0)$ for all $\theta \in (\theta_0, \theta_0 + \epsilon'')$. Take a nonlinear solution $\mathscr{S}$ and let

$$T \triangleq \inf_{\theta \in (\theta_0, \theta_0 + \epsilon'')} \frac{\mathscr{S}(\theta)}{\theta - \theta_0}.$$

By design, $T \in [w_1, 0]$. As $\frac{d\mathscr{S}}{d\theta}$ is uniquely determined by $\frac{\mathscr{S}(\theta)}{\theta - \theta_0}$ (recall $\frac{dS}{d\theta} = \frac{r(\theta - \theta_0)}{S} - k_1$) and $\frac{d^2 S}{d\theta^2} > 0$ for $\frac{S(\theta)}{\theta - \theta_0} \in (w_1, 0)$ and $\theta > \theta_0$,[18] for any z close to T and $z > T$, integral curve $\mathscr{S}$ crosses $Z(\theta) = z(\theta - \theta_0)$ on $(\theta_0, \theta_0 + \epsilon'')$ at most once and can only cross from below; that is, $\exists \epsilon_z > 0$ such that for all $\theta \in (\theta_0, \theta_0 + \epsilon_z)$, $\frac{\mathscr{S}(\theta)}{\theta - \theta_0} \in [T, z)$. Let $z \to T^+$. Then for $\theta \in (\theta_0, \theta_0 + \epsilon_z)$, $\frac{\mathscr{S}(\theta)}{\theta - \theta_0} \to T$ and $\frac{\mathscr{S}(\theta) - \mathscr{S}(\theta_0)}{\theta - \theta_0} \to T$. Plugging these two into

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[17]For any integral constant C, the integral curve infinitely swirls around the singularity $(\theta, S) = (\theta_0, 0)$ while approaching it. We discuss this case thoroughly in Online Appendix E.

[18]To see this, notice $\frac{d^2 S(\phi)}{d\phi^2} = \frac{d(\frac{dS}{d\phi})}{dS} \frac{dS}{d\phi} = r[\frac{1}{S} - \frac{\phi}{S^2}(\frac{r\phi}{S} - k_1)] = \frac{r\phi^2}{S^3}(w^2 + k_1 w - r)$ where $\phi \triangleq \theta - \theta_0$ and $w \triangleq \frac{S}{\phi}$.
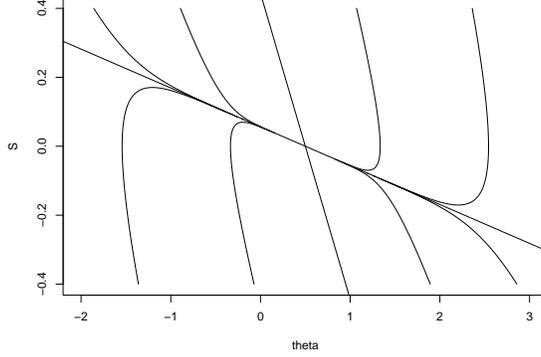
Figure 10: Solutions for $r < 0$, $\Delta > 0$
This figure is drawn for $k_1 = 1$, $k_2 = 0.8$, $b_1 = \delta = 0.5$, $b_2 = 0.1$.

differential equation (2), we have $T + k_1 = \frac{r}{T}$. And by $T \in [w_1, 0)$, $T = w_1$. Thus the right derivative exists and is w_1. A similar argument goes for the left derivative.

$\square$

Proof of Proposition 1. The necessity follows from Lemmas 1, 4, and 3. We next prove sufficiency.

We first calculate $\frac{\partial \mathscr{U}(\hat{\theta}; \theta)}{\partial \hat{\theta}}$ when the sender's strategy is continuous at $\hat{\theta}$:

$$
\begin{aligned}
\frac{\partial \mathscr{U}(\hat{\theta}; \theta)}{\partial \hat{\theta}} &= -2[\sigma(\hat{\theta}) - k_1\theta - b_1]\frac{d\sigma(\hat{\theta})}{d\hat{\theta}} - 2\delta(\hat{\theta} - k_2\theta - b2) \\
&= -2(\hat{\theta} - \theta)[k_1(S'(\hat{\theta}) + k_1) + \delta k_2] \\
&= -2(\hat{\theta} - \theta)[k_1\frac{r\hat{\theta} + \delta b_2}{S(\hat{\theta})} + \delta k_2] \\
&= (\hat{\theta} - \theta)\frac{\partial^2 \mathscr{U}(t; \hat{\theta})}{\partial t^2}\bigg|_{t=\hat{\theta}}
\end{aligned}
$$

The expression has the same sign as $\theta - \hat{\theta}$ by the second-order condition. Thus, whenever the sender's strategy is continuous at $\hat{\theta}$, the sender has an incentive to induce a slightly higher belief when $\theta > \hat{\theta}$ and a slightly lower belief when $\theta < \hat{\theta}$. Now suppose S is discontinuous at $\hat{\theta}$. By assumption, S jumps upward if $k_1 > 0$ and downward if $k_1 < 0$, with $|S|$ being continuous. Then $\mathscr{U}(\hat{\theta}+; \theta) - \mathscr{U}(\hat{\theta}-; \theta) > 0$ if $\theta > \hat{\theta}$, and $\mathscr{U}(\hat{\theta}+; \theta) - \mathscr{U}(\hat{\theta}-; \theta) < 0$ if $\theta < \hat{\theta}$. Therefore, the sender always benefits from inducing a belief that is closer to the true state θ. We therefore have incentive

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compatibility. $\qquad\square$

C   Omitted Proofs in Section 4

We first prove the Contraction and Expansion Lemmas, which help us identify the optimal incentive-compatible strategy in the proof of Theorem 2. The Off-Path-Belief Lemma specifies the off-path belief for the optimal incentive-compatible strategy. Then we can prove Theorem 2.

Definition 1. A strategy S is incentive compatible over $\mathscr{W}$ if $\mathscr{U}(\theta';\theta) \leq \mathscr{U}(\theta;\theta)$ for all $\theta, \theta' \in \mathscr{W}$.

Lemma 5. ***Contraction Transform****. Let Θ_1 and Θ_2 be two non-overlapping intervals partitioning Θ; that is, $\Theta = \Theta_1 \cup \Theta_2$, $\Theta_1 \cap \Theta_2 = \emptyset$, $\theta_1 < \theta_2$ for all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$. Suppose $S_1(\theta)$ is an incentive-compatible separating strategy. Suppose a separating strategy $S_2(\theta)$ is incentive compatible over Θ_2. If*

1. *$\forall \theta_2 \in \Theta_2$, $|S_2(\theta_2)| \leq |S_1(\theta_2)|$*

2. *$\forall \theta_1 \in \Theta_1$, $\forall \theta_2 \in \Theta_2$, $|S_1(\theta_2) + k_1(\theta_2 - \theta_1)| \leq |S_2(\theta_2) + k_1(\theta_2 - \theta_1)|$,*

then

$$S(\theta) \triangleq \begin{cases} S_1(\theta), & \text{if } \theta \in \Theta_1 \\ S_2(\theta), & \text{if } \theta \in \Theta_2 \end{cases}$$

is incentive compatible.

In other words, if an incentive-compatible separating strategy S_1 contracts over Θ_2 toward the bliss point of Θ_2 while moving away from the bliss point of Θ_1 (see Figure 11), it is still incentive compatible.

Proof of Lemma 5. Take a strategy S defined as above, and let σ denote its action. As S_1 is incentive compatible, a sender with state $\theta \in \Theta_1$ does not mimic any other state in Θ_1. As S_2 is incentive compatible over Θ_2, a sender with state $\theta \in \Theta_2$ does not mimic any other state in Θ_2. Let $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$. A sender with state θ_2 does
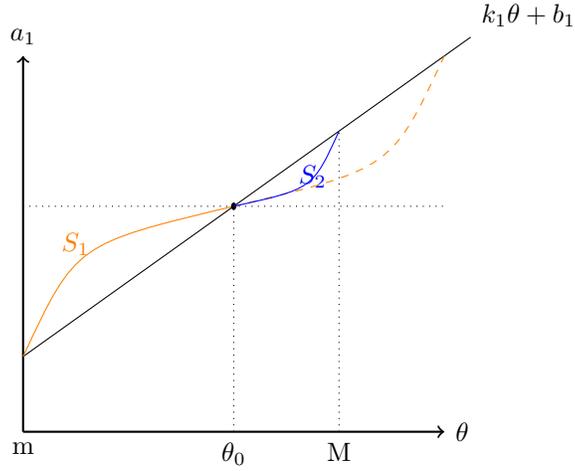
Figure 11: Contraction Transform

not mimic θ_1 because

$$
\begin{aligned}
\mathscr{U}(\theta_1; \theta_2) &\leq -(S_1(\theta_2))^2 - \delta(\theta_2 - k_2\theta_2 - b_2)^2 \\
&\leq -(S_2(\theta_2))^2 - \delta(\theta_2 - k_2\theta_2 - b_2)^2 \\
&= -(S(\theta_2))^2 - \delta(\theta_2 - k_2\theta_2 - b_2)^2 \\
&= \mathscr{U}(\theta_2; \theta_2).
\end{aligned}
$$

The first inequality is true because S_1 is incentive compatible, and the second inequality is true by assumption.

The sender with state θ_1 does not mimic θ_2, because

$$
\begin{aligned}
\mathscr{U}(\theta_2; \theta_1) &= -(\sigma(\theta_2) - k_1\theta_1 - b_1)^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2 \\
&= -(S(\theta_2) + k_1(\theta_2 - \theta_1))^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2 \\
&= -(S_2(\theta_2) + k_1(\theta_2 - \theta_1))^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2 \\
&\leq -(S_1(\theta_2) + k_1(\theta_2 - \theta_1))^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2 \\
&\leq -(S_1(\theta_1))^2 - \delta(\theta_1 - k_2\theta_1 - b_2)^2 \\
&= \mathscr{U}(\theta_1; \theta_1).
\end{aligned}
$$

The first inequality is true by assumption, and the second inequality is true because S_1 is incentive compatible. $\qquad \square$

For the following lemma, we only state the case of positive S.

Lemma 6. _Monotone Expansion_. Let $\theta^0 \in \Theta$ and $\theta^0 \neq \theta_0$. Suppose $q\theta^0 + n \geq 0$. Let $S(\theta; q, n, p)$ denote the solution for

$$\frac{dS}{d\theta} = \frac{q\theta + n}{S} + p, \quad S \geq 0$$

with initial value $S(\theta^0; q, n, p) = 0$.[19] For any $\theta \geq \theta^0$, $S(\theta; q, n, p)$ monotonically increases in n and p, increases in q if $\theta > 0$, and decreases in q if $\theta < 0$.

Proof of Lemma 6. It suffices to prove the monotonicity in p. Let $p_1 \leq p_2$. We show $S(\theta; q, n, p_1) \leq S(\theta; q, n, p_2)$ for $\theta \geq \theta^0$.

By way of contradiction, suppose $\exists \theta^* > \theta^0$ such that $S(\theta^*; q, n, p_1) > S(\theta^*; q, n, p_2)$. Because $S(\theta^0; q, n, p_1) = S(\theta^0; q, n, p_2) = 0$ and

$$S(\theta; q, n, p) = \int_{\theta^0}^{\theta} \frac{\partial S(\theta; q, n, p)}{\partial \theta} d\theta,$$

there must exist $\tilde{\theta} \in (\theta^0, \theta^*)$ such that

$$S(\tilde{\theta}; q, n, p_1) > S(\tilde{\theta}; q, n, p_2)$$

and

$$\left. \frac{\partial S(\theta; q, n, p_1)}{\partial \theta} \right|_{\theta=\tilde{\theta}} > \left. \frac{\partial S(\theta; q, n, p_2)}{\partial \theta} \right|_{\theta=\tilde{\theta}},$$

contradicting

$$\begin{aligned}
\left. \frac{\partial S(\theta; q, n, p_1)}{\partial \theta} \right|_{\theta=\tilde{\theta}} &= \frac{q\theta + n}{S(\tilde{\theta}; q, n, p_1)} + p_1 \\
&< \frac{q\theta + n}{S(\tilde{\theta}; q, n, p_2)} + p_1 \\
&\leq \frac{q\theta + n}{S(\tilde{\theta}; q, n, p_2)} + p_2 \\
&= \left. \frac{\partial S(\theta; q, n, p_2)}{\partial \theta} \right|_{\theta=\tilde{\theta}}.
\end{aligned}$$

$\square$

Lemma 7. _Off-Path Belief_. Let $A_1^S \triangleq \{a_1^S(\theta) | \theta \in \Theta\}$. If $A_1^S \subset A_1$, a continuous

---

[19]When $\theta^0 \neq \theta_0$, the differential equation with the initial value has a unique solution.

incentive-compatible strategy $\sigma(\theta)$ and degenerate off-path belief

$$\hat{\theta}(a_1) = \begin{cases} \underline{\theta}, & \text{if } a_1 < \min_{\theta \in \Theta} \sigma(\theta) \\ \overline{\theta}, & \text{if } a_1 > \max_{\theta \in \Theta} \sigma(\theta). \end{cases}$$

Here, $\underline{\theta} \in \arg\min_{\theta \in \Theta} \sigma(\theta)$, $\overline{\theta} \in \arg\max_{\theta \in \Theta} \sigma(\theta)$ form a perfect Bayesian equilibrium.

Proof of Lemma 7. If the sender takes action $a_1 < \min_{\theta \in \Theta} \sigma(\theta)$ given state θ, the receiver believes $\hat{\theta} = \underline{\theta}$. Hence, the sender's payoff

$$\begin{aligned} V(\theta, \underline{\theta}, a_1) &= -(a_1 - k_1\theta - b_1)^2 - \delta(\underline{\theta} - k_2\theta - b_2)^2 \\ &< -(\min_{\theta \in \Theta} \sigma(\theta) - k_1\theta - b_1)^2 - \delta(\underline{\theta} - k_2\theta - b_2)^2 \\ &= \mathscr{U}(\underline{\theta}; \theta) \end{aligned}$$

because $A_1^S \subset A_1$. That is, this action is strictly worse than mimicking state $\underline{\theta}$. Therefore, this deviation is not profitable for the sender.

On the other hand, if the sender takes action $a_1 > \max_{\theta \in \Theta} \sigma(\theta)$ at state θ, the receiver believes $\theta = \overline{\theta}$. The sender's payoff

$$\begin{aligned} V(\theta, \overline{\theta}, a_1) &= -(a_1 - k_1\theta - b_1)^2 - \delta(\overline{\theta} - k_2\theta - b_2)^2 \\ &< -(\max_{\theta \in \Theta} \sigma(\theta) - k_1\theta - b_1)^2 - \delta(\overline{\theta} - k_2\theta - b_2)^2 \\ &= \mathscr{U}(\overline{\theta}; \theta). \end{aligned}$$

This action is strictly worse than mimicking state $\overline{\theta}$. $\square$

Proof of Theorem 2. Throughout this proof, it is very helpful to keep in mind the figures of integral curves. And we only need to consider the case of $k_1 > 0$ since the cases of $k_1 > 0$ and $k_1 < 0$ are symmetric around the θ-axis by differential equation (2). Let S^* denote the optimal incentive-compatible strategy. Let $\tilde{S}$ denote any discontinuous incentive-compatible strategy and $\tilde{\theta}$ denote any discontinuous point.

As

$$\frac{d^2 S(\phi)}{d\phi^2} = \frac{d(\frac{dS}{d\phi})}{dS}\frac{dS}{d\phi} = r\left(\frac{1}{S} - \frac{\phi}{S^2}\frac{dS}{d\phi}\right),$$

the second-order condition can be reduced to

$$
\frac{\partial^2 \mathscr{U}(\theta';\theta)}{\partial \theta'^2}\bigg|_{\theta'=\theta} = -2\left(\frac{dS(\theta)}{d\theta} + k_1\right)^2 - 2S\frac{d^2 S(\theta)}{d\theta^2} - 2\delta
$$

$$
= -2\delta\left[k_2 + k_1\frac{(k_2 - 1)\theta + b_2}{S}\right]
$$

$$
\leq 0.
$$

Notice that any linear solution of (2) satisfies the second-order condition trivially, as $-2\left(\frac{d\sigma(\theta)}{d\theta}\right)^2 - 2\delta \leq 0$.

Case 1: $r = 0$

We plot the solutions in Figure 12. The SOC is $\delta\left[\frac{k_1 b_2}{S} + 1\right] \geq 0$. We only analyze the case of $b_2 > 0$[20] since the case of $b_2 < 0$ can be addressed centro-symmetrically. When $b_2 > 0$, the SOC requires $S \geq 0$ or $S \leq -k_1 b_2$. There is one strategy crossing
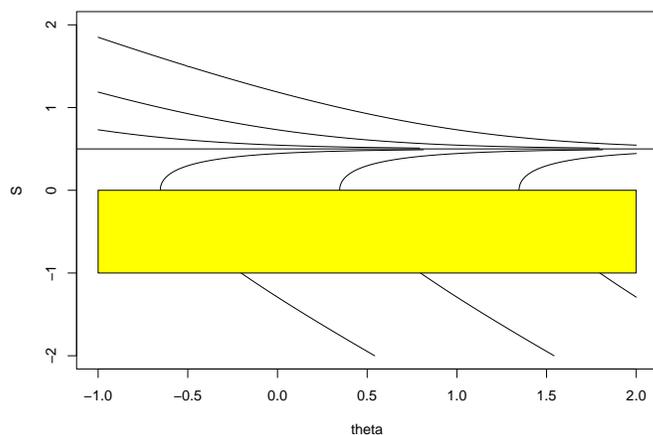

Figure 12: Solutions for $r = 0$
The yellow region is removed by the SOC.

$(\theta, S) = (m, 0)$,

$$
k_1^2(\theta - m) + k_1 S^* + \delta b_2 \ln(\delta b_2 - k_1 S^*) = \delta b_2 \ln(\delta b_2) \qquad (S^* \geq 0).
$$

It is uniformly closest to sender's bliss point $S = 0$ among all incentive-compatible strategies $S \geq 0$. Next we show it dominates any incentive-compatible strategy

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[20]When $b_2 = 0$, the optimal solution is trivially $S = 0$.

$S < 0$. For a solution $k_1^2\theta + k_1 S + \delta b_2 \ln(\delta b_2 - k_1 S) = C$, let $S^+(\theta; C)$, $S^-(\theta; C)$ denote the positive branch ($S \geq 0$) and the negative one ($S \leq 0$). By the Monotone Expansion Lemma 6, $|S^+(\theta; C)| \leq |S^-(\theta; C)|$. Thus, S^* dominates all incentive-compatible strategies $S \leq 0$ defined over Θ. Next we show S^* dominates any discontinuous incentive-compatible strategy $\tilde{S}$. Although the SOC does not bind S^*, it binds the initial condition of any discontinuous incentive-compatible strategy $\tilde{S}$ such that $\tilde{S}(0) \leq -k_1 b_2$. Let $S^-(\theta; \delta b_2 \ln(\delta b_2) + k_1^2 m)$ denote the negative branch of the integral curve on which S^* lies. For $\theta \in [m, \tilde{\theta}]$,

$$|\tilde{S}(\theta)| \geq |S^-(\theta; \delta b_2 \ln(\delta b_2) + k_1^2 m)| \geq |S^*(\theta)|,$$

where the second inequality is true by the Monotone Expansion Lemma. Because $|\tilde{S}(\tilde{\theta})| \geq |S^*(\tilde{\theta})|$ and $|S|$ is continuous,

$$|\tilde{S}(\theta)| \geq |S^*(\theta)|, \quad \theta \in [\tilde{\theta}, M].$$

$\tilde{S}$ is uniformly dominated by S^*.

Thus, the optimal incentive-compatible strategy is unique and is S^*, which is continuous. By $\frac{d\sigma}{d\theta} = \frac{\delta b_2}{S} > 0$, it is monotonic. Moreover, by $k_1 > 0$,

$$A_1^S \subset A_1.$$

Thus, it supports a perfect Bayesian equilibrium by Lemma 7.

Case 2: $r > 0$

In this case, $\Delta > 0$. In Figure 13, we plot the solution for $r > 0$.

Suppose $\theta_0 \notin \Theta$. It suffices to consider $\theta_0 < m$. There is one solution crossing $(\theta, S) = (m, 0)$ among all solutions (9):

$$\frac{(w_1\phi - S^*)^{w_1}}{(S^* - w_2\phi)^{w_2}} = \frac{[w_1(m - \theta_0)]^{w_1}}{[w_2(\theta_0 - m)]^{w_2}} \qquad (S^* \geq 0, \ \phi \in [m - \theta_0, M - \theta_0])$$

It satisfies the SOC and is uniformly closest to the sender's bliss point $S = 0$ among all incentive-compatible strategies $S \geq 0$. By the Monotone Expansion Lemma 6, S^* dominates any incentive-compatible strategy $S < 0$. By the same argument as in case 1, S^* dominates any discontinuous incentive-compatible strategies. Thus, the optimal incentive-compatible strategy is unique and is S^*, which is continuous and
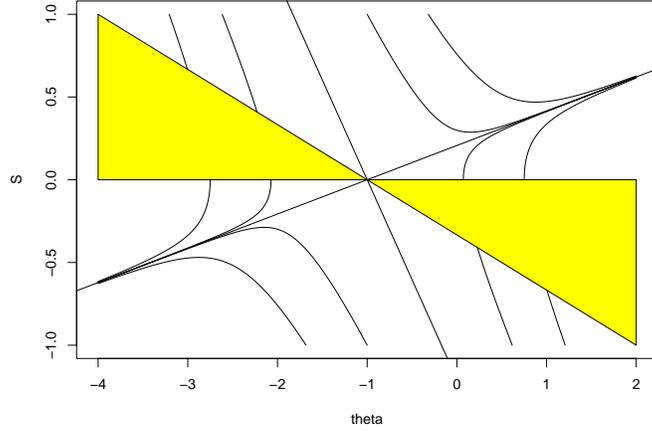
Figure 13: Solutions for $r > 0$
The yellow shadow is the region removed by the SOC. This figure is drawn for $k_1 = 1$, $k_2 = 1.5$, $b_1 = b_2 = \delta = 0.5$.

nonlinear. Because $\frac{d\sigma}{d\theta} = \frac{r\phi}{S^*} > 0$, it is monotonic. Moreover, it supports a perfect Bayesian equilibrium by Lemma 7.

Suppose $\theta_0 \in \Theta$. Branches $\frac{(w_1\phi - S)^{w_1}}{(S - w_2\phi)^{w_2}} = C$, $\frac{(S - w_1\phi)^{w_1}}{(w_2\phi - S)^{w_2}} = C$ cannot be supported on Θ. In addition, Branches $\frac{(S - w_1\phi)^{w_1}}{(S - w_2\phi)^{w_2}} = C$, $\frac{(w_1\phi - S)^{w_1}}{(w_2\phi - S)^{w_2}} = C$, $S = w_2\phi$ are uniformly dominated by $S^* = w_1\phi$. Next we show S^* uniformly dominates any discontinuous incentive-compatible strategy $\tilde{S}$. If $w_1(m - \theta_0) < \tilde{S}(m) < w_2(m - \theta_0)$, we must have $\tilde{\theta} < \theta_0$,

$$|\tilde{S}(\tilde{\theta})| < w_2(\tilde{\theta} - \theta_0)$$

because $w_1 > 0 > w_2$ and $|w_1| \leq |w_2|$. Then, $\tilde{S}$ cannot be supported on Θ. If $\tilde{S}(m) < w_1(m - \theta_0)$, $\tilde{S}$ either cannot be supported on Θ if $|\tilde{S}(\tilde{\theta})| < |w_2(\tilde{\theta} - \theta_0)|$ or is uniformly dominated by $S^* = w_1(\theta - \theta_0)$ if $|\tilde{S}(\tilde{\theta})| > |w_2(\tilde{\theta} - \theta_0)|$.

Thus, the optimal incentive-compatible strategy is unique and is S^*, which is continuous, monotonic, and linear. Moreover, as $w_1 > 0$ and $k_1 > 0$, $A_1^S \subset A_1$. Thus, S^* supports a perfect Bayesian equilibrium by Lemma 7.

Case 3: $r < 0$

We further categorize this case in terms of Δ.

Case 3.1: $r < 0$, $\Delta > 0$

In Figure 14, we plot the solution for $r < 0$, $\Delta > 0$.
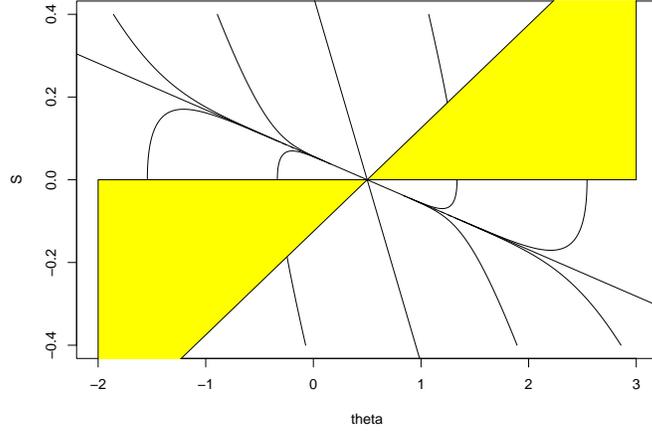
Figure 14: Solutions for $r < 0$, $\Delta > 0$
The yellow shadow identifies the region removed by the SOC. This figure is drawn for
$k_1 = 1$, $k_2 = 0.8$, $b_1 = \delta = 0.5$, $b_2 = 0.1$.

Suppose $\theta_0 \notin \Theta$. If $\theta_0 \geq M$, there is one solution crossing $(\theta, S) = (m, 0)$:

$$\frac{(w_1\phi - S^*)^{w_1}}{(w_2\phi - S^*)^{w_2}} = \frac{[w_1(m - \theta_0)]^{w_1}}{[w_2(m - \theta_0)]^{w_2}} \qquad (S^* \geq 0, \ \phi \in [m - \theta_0, M - \theta_0]) \qquad (13)$$

It satisfies the SOC and is uniformly closest to the sender's bliss point $S = 0$ among all $S \geq 0$ incentive-compatible strategies. By the monotone expansion property, it dominates any incentive-compatible strategy with $S < 0$. By the same argument as in case 1, S^* dominates any discontinuous incentive-compatible strategies. Thus, S^* is the unique optimal incentive-compatible strategy, and it is continuous and nonlinear. As $\frac{d\sigma}{d\theta} = \frac{r\phi}{S^*} > 0$, S^* is monotonic. Moreover, S^* supports a perfect Bayesian equilibrium by Lemma 7. If $\theta_0 \leq m$, the same argument applies and the optimal one crosses $(\theta, S) = (M, 0)$:

$$\frac{(S^* - w_1\phi)^{w_1}}{(S^* - w_2\phi)^{w_2}} = \frac{(-w_1(M - \theta_0))^{w_1}}{(-w_2(M - \theta_0))^{w_2}} \qquad (S^* \leq 0, \ \phi \in [m - \theta_0, M - \theta_0]) \qquad (14)$$

Suppose $\theta_0 \in \Theta$. By the argument above, for $\theta \in [m, \theta_0]$, solution (13) is optimal; for $\theta \in [\theta_0, M]$, solution (14) is optimal. Nevertheless, since (13) and (14) are not necessarily[21] on the same integral curve, we need to check whether the combination

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[21]They are on the same integral curve if and only if $\theta_0 = \frac{m+M}{2}$.

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S^*: (13)+(14) is incentive compatible. By the centrosymmetry of DE (5), any integral curve crossing $(\phi, S) = (0, 0)$ is centrosymmetic around $(\phi, S) = (0, 0)$. By DE (5),

$$\frac{d\sigma(\theta)}{d\theta} = \frac{dS^*}{d\phi} + k_1 = \frac{\delta(k_2 - 1)\phi}{S^*} > 0$$

on the combination (13)+(14). Thus, by Lemma 5, the combination is a contraction transform (see Figure 11) of the integral curve (13) if $\theta_0 > \frac{m+M}{2}$, (14) if $\theta_0 < \frac{m+M}{2}$. Therefore, the combination S^* is incentive compatible and dominates any continuous incentive-compatible strategy. Next we show S^* dominates any discontinuous incentive-compatible strategy $\tilde{S}$.

By the Monotone Expansion Lemma and the second-order condition,

$$|\tilde{S}(\tilde{\theta})| \geq |S^*(\tilde{\theta})|.$$

If $|\tilde{S}(\tilde{\theta})| \leq |w_2(\tilde{\theta} - \theta_0)|$, $\tilde{S}$ is uniformly dominated by S^*; if $|\tilde{S}(\tilde{\theta})| > |w_2(\tilde{\theta} - \theta_0)|$, $\tilde{S}$ is either undefined for the entire Θ or uniformly dominated by S^* by the Monotone Expansion Lemma.

Therefore, S^* is the unique optimal incentive-compatible strategy and is nonlinear, monotonic, and continuous. By Lemma 7, it supports a perfect Bayesian equilibrium.

The analysis for the remaining cases below is similar.

Case 3.2: $r < 0$, $\Delta = 0$ (See Figure 15)

If $\theta_0 > M$, the unique optimal solution crosses $(\theta, S) = (m, 0)$:

$$\ln(w_1\phi - S^*) - \frac{w_1\phi}{S^* - w_1\phi} = \ln(w_1(m - \theta_0)) + 1 \qquad (S^* \geq 0, \ \phi \in [m - \theta_0, M - \theta_0]) \ (15)$$

The solution satisfies the SOC and is continuous and nonlinear. It is monotonic by $\frac{d\sigma}{d\theta} = \frac{r\phi}{S^*} > 0$. By Lemma 7, it supports a perfect Bayesian equilibrium. If $\theta_0 < m$, the unique optimal strategy crosses $(\theta, S) = (M, 0)$:

$$\ln(S^* - w_1\phi) - \frac{w_1\phi}{S^* - w_1\phi} = \ln(-w_1(M - \theta_0)) + 1 \qquad (S^* \leq 0, \ \phi \in [m - \theta_0, M - \theta_0])$$

$$(16)$$

This satisfies the SOC and is continuous and nonlinear. It is monotonic because $\frac{d\sigma}{d\theta} = \frac{r\phi}{S^*} > 0$. By Lemma 7, it supports a perfect Bayesian equilibrium. If $\theta_0 \in \Theta$, by the same argument as in case 3.2, the combination (15)+(16) is the unique optimal
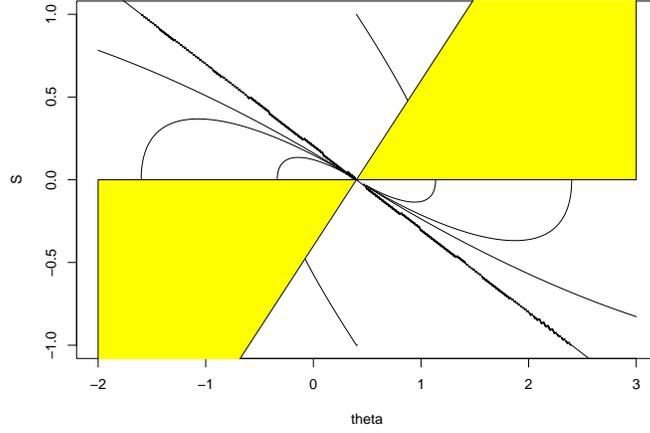
Figure 15: Solutions for $r < 0$, $\Delta = 0$
The yellow region is removed by the SOC. This figure is drawn for $k_1 = 1$, $k_2 = 0.5$, $b_1 = 0.5$, $b_2 = 0.2$, $\delta = 0.4$.

incentive-compatible strategy, which is nonlinear, continuous, and monotonic and supports a perfect Bayesian equilibrium.

$\square$

Proof of Proposition 5. If the sender's strategy is linear, we can directly check how $|w_1|$ varies with δ by (7) and (8).

Now suppose the sender's strategy in Riley equilibrium is nonlinear. If $r = 0$, we can apply the Monotone Expansion Lemma to

$$\frac{dS}{d\theta} = \frac{\delta b_2}{S} - k_1$$

with $\theta^0 = m$ or $\theta^0 = M$. If $r \neq 0$, we can similarly apply the Monotone Expansion Lemma to

$$\frac{dS}{d\phi} = \frac{r\phi}{S} - k_1$$
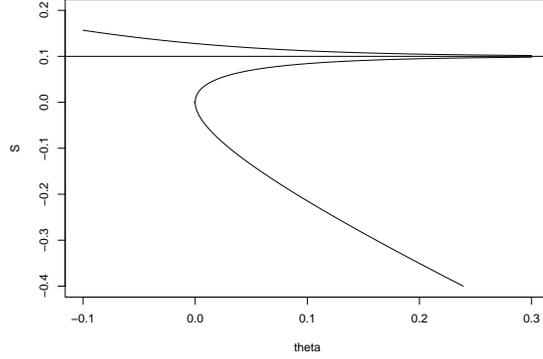
with $\theta^0 = m$ or $\theta^0 = M$. $\square$

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Figure 16: Delegation $S(\theta)$
This figure is drawn for $k_1 = k_2 = 1$, $b_1 = b_2 = 0.2$, $\delta = 0.5$.

D   Omitted Proofs in Section 5

Proof of Proposition 3. The proof follows directly by checking each solution in the proof of Theorem 2 and taking $k_1 = 1$ and $b_1 = 0$.   □

Proof of Proposition 4. By the proof of Theorem 2, any incentive-compatible strategy belongs to one of the following two cases:
(1) $S(\theta)$ is continuous (Figure 16):

$$\theta + S + \delta b \ln|S - \delta b| = C, \quad (S \geq 0 \text{ or } S \leq -b)$$

or

$$S = \delta b \tag{17}$$

(2) $S(\theta)$ is discontinuous at some $\tilde{\theta} \in (0, 1)$ and satisfies the following conditions:

$$\theta + S + \delta b \ln(\delta b - S) = C \quad (S \leq -b)$$

for $\theta < \tilde{\theta}$, and

$$\theta + S + \delta b \ln(S - \delta b) = C' \quad (S > \delta b)$$

for $\theta > \tilde{\theta}$, where C and C' are chosen such that $-S(\tilde{\theta}-) = S(\tilde{\theta}+)$.

By Lemma 9, all discontinuous strategies are Pareto dominated by linear strategy

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(17). By Lemma 8, continuous decreasing strategies

$$\theta + S + \delta b \ln(\delta b - S) = C, \quad S \leq -b \tag{18}$$

cannot be part of any perfect Bayesian equilibrium.

Among all increasing continuous strategies, the (agent's) optimal incentive-compatible strategy is closest to the principal's ideal point.

$\square$

Lemma 8. *Strategy (18) cannot be part of a perfect Bayesian equilibrium.*

Proof of Lemma 8. Since
$$\frac{dS}{d\theta} = \frac{\delta b}{S} - 1 < -1,$$
we have
$$S(\theta) < -b - \theta.$$

Since $\sigma(\theta) \leq 0$ on the equilibrium path for all θ, we consider a deviation to $a_1 = \theta + b$ at some state $\theta > 0$.

If the off-path belief assigns probability 1 to $\theta = 0$ when the agent deviates to $a_1 = \theta + b$, the agent's payoff is

$$V(\theta, 0, \theta + b) = -\delta(\theta + b)^2 \geq -(\theta + b)^2 > -S^2(\theta) > -S^2(\theta) - \delta b^2 = V(\theta, \theta, \sigma(\theta)),$$

a profitable deviation regardless of θ.

If the off-path belief assigns probability 1 to $\theta = 1$ when the agent deviates to $a_1 = \theta + b$, the agent's payoff is

$$V(\theta, 1, \theta + b) = -\delta(1 - \theta - b)^2.$$

When state $\theta \geq \frac{1}{2} - b$,

$$V(\theta, 1, \theta + b) = -\delta(1 - \theta - b)^2 \geq -\delta(\theta + b)^2 > -S^2(\theta) > V(\theta, \theta, \sigma(\theta)),$$

a profitable deviation whenever $\theta \geq \frac{1}{2} - b$.

For other possible off-path beliefs, the principal will take a second-period action within the interval $(0, 1)$. Hence the agent's payoff is bounded below by $\min\{-\delta(\theta +$

$b)^2, -\delta(1-\theta-b)^2\}$. Therefore, there is always a profitable deviation when $\theta \geq \frac{1}{2} - b$. $\qquad\square$

Lemma 9. *Any discontinuous incentive-compatible strategy in a perfect Bayesian equilibrium is Pareto dominated by the linear strategy $S = \delta b$.*
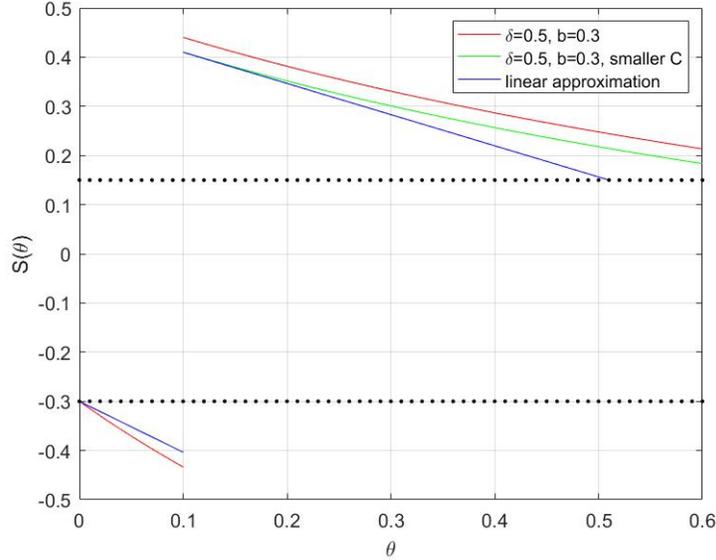

Figure 17: Discontinuous Colution

Proof of Lemma 9. Compared with the discontinuous solution, the linear special solution is uniformly closer to the agent's bliss point. Thus, it suffices to prove that the linear special solution dominates from the principal's perspective.

This proof proceeds as follows (see Figure 17): 1. Move the left area $S \leq -b$ closer to the principal's bliss point $S = -b$ to form an upper bound to her payoff through linear approximation over interval $[0, \tilde\theta]$. 2. Move the right area $S > \delta b$ to the left (decrease the constant C_1) and then reach an upper bound to the principal's payoff through linear approximation over the interval $[\tilde\theta, \tilde\theta + D]$ (D is defined as the size of interval when the linear approximation is above $S = \delta b$). Notice that the movement in the second step relies on the first step since $|S|$ is continuous at $\tilde\theta$. 3. Assume $\tilde\theta + D \leq 1$. Prove that the upper bound of the principal's payoff by linear approximation is lower than that of the special linear solution, $-(b + \delta b)^2$. 4. Prove that $\tilde\theta + D \leq 1$; otherwise this solution could not be a perfect Bayesian equilibrium.

Step 1. Define the principal's payoff over $[0, \tilde{\theta}]$ as u_1 and her payoff over $[0, \tilde{\theta}]$ of linear approximation as U_1. For $0 \leq \theta \leq \tilde{\theta}$, $S \leq -b$,

$$\frac{dS}{d\theta} = \frac{\delta b}{S} - 1 \geq -(1 + \delta)$$

$$S(\tilde{\theta}^-) > -b - (1 + \delta)\tilde{\theta}.$$

Since $\frac{d^2 S}{d\theta^2} \geq 0$,

$$\frac{dS}{d\theta} < -[1 + \frac{\delta b}{b + (1 + \delta)\tilde{\theta}}].$$

Define K as

$$K = 1 + \frac{\delta b}{b + (1 + \delta)\tilde{\theta}}$$

$$S(\theta) = S(0) + \int_0^\theta \frac{dS(t)}{dt} dt < -b - K\theta$$

$$u_1 = - \int_0^{\tilde{\theta}} [S(\theta) + b]^2 d\theta < - \int_0^{\tilde{\theta}} (K\theta)^2 d\theta = -\frac{1}{3} K^2 \tilde{\theta}^3 = U_1$$

$$S(\tilde{\theta}^-) = S(0) + \int_0^{\tilde{\theta}} \frac{dS(\theta)}{d\theta} d\theta < -b - K\tilde{\theta}.$$

Define S_0 as

$$S_0 = b + K\tilde{\theta}.$$

Step 2. Denote the principal's payoff over $[\tilde{\theta}, 1]$ as u_2 and her payoff over $[\tilde{\theta}, 1]$ of linear approximation as U_2. Denote the solution passing through $(\tilde{\theta}^+, S(\tilde{\theta}^+))$ as $\theta + S(\theta, C_1) + \delta b \ln[S(\theta, C_1) - \delta b] = C_1$, and the solution passing through $(\tilde{\theta}^+, S_0)$ as $\theta + S(\theta, C_1^*) + \delta b \ln[S(\theta, C_1^*) - \delta b] = C_1^*$.

For $\tilde{\theta} \leq \theta \leq 1$, $S > \delta b$,

$$\frac{\partial S(\theta, C)}{\partial C} = -\frac{\partial S(\theta, C)}{\partial \theta} > 0.$$

Since $S(\tilde{\theta}^+) = -S(\tilde{\theta}^-) > S_0$,

$$C_1 > C_1^*$$

$$S(\theta, C_1) > S(\theta, C_1^*).$$

Since $\frac{\partial S^2(\theta, C_1^*)}{\partial \theta^2} > 0$

$$S(\theta, C_1^*) \geq S_0 + (\theta - \tilde{\theta}) \frac{\partial S(t, C_1^*)}{\partial t}\Big|_{t=\tilde{\theta}+} = S_0 + (\theta - \tilde{\theta})(\frac{\delta b}{S_0} - 1).$$

Define k as

$$k = -\frac{\partial S(t, C_1^*)}{\partial t}\Big|_{t=\tilde{\theta}+} = 1 - \frac{\delta b}{S_0}.$$

Define D as the size of interval when the linear approximation is above $S = \delta b$

$$S_0 + D(\frac{\delta b}{S_0} - 1) = \delta b$$

$$D = S_0.$$

Therefore, for $\theta \in [\tilde{\theta}, \tilde{\theta} + D]$,

$$S(\theta, C_1) > S(\theta, C_1^*) \geq S_0 - k(\theta - \tilde{\theta}) \geq \delta b,$$

while $S(\theta, C_1) > \delta b$ for $\theta \in [\tilde{\theta} + D, 1]$. Here we assume $\tilde{\theta} + D \leq 1$, and we prove that this assumption is guaranteed in step 3.

$$\begin{aligned}
u_2 &= -\int_{\tilde{\theta}}^1 [S(\theta, C_1) + b]^2 d\theta \\
&< -\int_{\tilde{\theta}}^1 [S(\theta, C_1^*) + b]^2 d\theta \\
&< -\int_{\tilde{\theta}}^{\tilde{\theta}+D} [S_0 - k(\theta - \tilde{\theta}) + b]^2 d\theta - \int_{\tilde{\theta}+D}^1 (\delta b + b)^2 d\theta \\
&= -\int_0^D [kt + (b + \delta b)]^2 dt - (b + \delta b)^2 (1 - \tilde{\theta} - D) \\
&= U_2
\end{aligned}$$

Step 3. To prove $u_1 + u_2 < -(b + \delta b)^2$, it suffices to prove

$$U_1 + U_2 < -(b + \delta b)^2. \tag{19}$$

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Define $p = b + \delta b$, $q = b - \delta b$. (19) becomes

$$-\frac{1}{3}K^2\tilde{\theta}^3 - pkD^2 - k^2\frac{D^3}{3} < -p^2\tilde{\theta}.$$

It would be sufficient if we could prove that

$$pkD^2 \geq p^2\tilde{\theta}.$$

With the definitions of k, D, and S_0,

$$S_0(S_0 - \delta b) \geq p\tilde{\theta}$$

$$(b + K\tilde{\theta})(b + K\tilde{\theta} - \delta b) \geq p\tilde{\theta}. \tag{20}$$

For inequality (20), define $G(\tilde{\theta})$ as

$$G(\tilde{\theta}) = \text{LHS} - \text{RHS}.$$

To prove (20), it suffices to prove that $G(\tilde{\theta}) \geq 0$. Since $K = 1 + \frac{\delta b}{b + (1+\delta)\tilde{\theta}} > 1$,

$$G(\tilde{\theta}) > (\tilde{\theta} + b)(\tilde{\theta} + q) - p\tilde{\theta} = \tilde{\theta}^2 + b(1 - 2\delta)\tilde{\theta} + bq.$$

However, if $\delta \leq 0.5$, then $G(\tilde{\theta}) > 0$, which concludes our proof. So we focus on the case of

$$\delta > 0.5 \tag{21}$$

from now on.

$$G(\tilde{\theta}) = K^2\tilde{\theta}^2 + [K(q + b) - p]\tilde{\theta} + qb$$

$$G(0) = bq \geq 0$$

To prove $G(\tilde{\theta}) \geq 0$, it is sufficient to prove

$$G'(\tilde{\theta}) \geq 0$$

$$G'(\tilde{\theta}) = 2\tilde{\theta}K^2 + K(b + q) - p + 2\tilde{\theta}^2KK' + (b + q)\tilde{\theta}K'.$$

By $K(\tilde{\theta}) = 1 + \frac{\delta b}{b+(1+\delta)\tilde{\theta}}$, we can find its derivative with regard to $\tilde{\theta}$:

$$K'(\tilde{\theta}) = -\delta b \frac{1+\delta}{[b+(1+\delta)\tilde{\theta}]^2}$$

$$K''(\tilde{\theta}) = 2\delta b \frac{(1+\delta)^2}{[b+(1+\delta)\tilde{\theta}]^3}$$

Since $G'(0) = K(b+q) - p = (1+\delta)(b+q) - p = (1+\delta)q \geq 0$, to arrive at $G'(\tilde{\theta}) \geq 0$, it is sufficient to prove

$$G''(\tilde{\theta}) \geq 0 \tag{22}$$

$$
\begin{aligned}
G''(\tilde{\theta}) =& 4\tilde{\theta}KK' + 2K^2 + (b+q)K' + 2(2\tilde{\theta}KK' + \tilde{\theta}^2 K'K' + \tilde{\theta}^2 KK'') + (b+q)(K' + \tilde{\theta}K'') \\
=& 8\tilde{\theta}KK' + 2K^2 + 2(b+q)K' + 2\tilde{\theta}^2 K'K' + 2\tilde{\theta}^2 KK'' + (b+q)\tilde{\theta}K''.
\end{aligned}
$$

To prove (22), because $2\tilde{\theta}^2 K'K' + 2\tilde{\theta}^2 KK'' > 0$, it is sufficient to prove

$$4\tilde{\theta}KK' + K^2 + (b+q)K' + \frac{b+q}{2}\tilde{\theta}K'' \geq 0$$

$$K^2 + \frac{b+q}{2}\tilde{\theta}K'' \geq [4\tilde{\theta}K + (b+q)] \cdot |K'|.$$

Plugging in $K(\tilde{\theta})$, $K'(\tilde{\theta})$, and $K''(\tilde{\theta})$, it becomes

$$(1+\delta)^2(b+\tilde{\theta})^2 + \frac{b+q}{2}\tilde{\theta}\frac{2\delta b(1+\delta)^2}{b+(1+\delta)\tilde{\theta}} \geq [4\tilde{\theta}K + (b+q)]\delta b(1+\delta)$$

$$(1+\delta)(b+\tilde{\theta})^2 + \tilde{\theta}\frac{(b+q)\delta b(1+\delta)}{b+(1+\delta)\tilde{\theta}} \geq [4\tilde{\theta}K + (b+q)]\delta b$$

$$(1+\delta)\tilde{\theta}^2 + \tilde{\theta}b[2(1+\delta) + \frac{(b+q)\delta(1+\delta)}{b+(1+\delta)\tilde{\theta}} - 4K\delta] + (1+\delta)b^2 - \delta b^2(2-\delta) \geq 0$$

$$(1+\delta)\tilde{\theta}^2 + \tilde{\theta}b[2(1+\delta) + \frac{\delta(b+q-4b-4\tilde{\theta})(1+\delta)}{b+(1+\delta)\tilde{\theta}}] + (1+\delta^2-\delta)b^2 \geq 0$$

$$(1+\delta)\tilde{\theta}^2 + \tilde{\theta}b(1+\delta)[2 - \delta\frac{(2+\delta)b+4\tilde{\theta}}{b+(1+\delta)\tilde{\theta}}] + (1+\delta^2-\delta)b^2 \geq 0. \tag{23}$$

Analyze the term in the square bracket $\frac{(2+\delta)b+4\tilde{\theta}}{b+(1+\delta)\tilde{\theta}}$.

If $\frac{2+\delta}{1} \leq \frac{4}{1+\delta}$, then $\delta \leq \delta^* \approx 0.56$

$$[2 - \delta \frac{(2+\delta)b + 4\tilde{\theta}}{b + (1+\delta)\tilde{\theta}}] \geq 2 - \delta \frac{4}{1+\delta} \geq 2 - \delta \frac{4}{1+\frac{1}{2}} \geq 2 - \frac{8\delta^*}{3} > 0,$$

where the second inequality is ensured by (21). Therefore, this concludes our proof.

If $\frac{2+\delta}{1} > \frac{4}{1+\delta}$, $\delta > \delta^* \approx 0.56$ and $[2 - \delta \frac{(2+\delta)b+4\tilde{\theta}}{b+(1+\delta)\tilde{\theta}}] > 2 - \delta(2+\delta)$. For (23),

$$LHS > (1+\delta)\tilde{\theta}^2 + \tilde{\theta}b(1+\delta)[2 - \delta(2+\delta)] + (1 + \delta^2 - \delta)b^2.$$

Thus, it suffices to prove

$$(1+\delta)\tilde{\theta}^2 + \tilde{\theta}b(1+\delta)(2 - 2\delta - \delta^2) + (1 + \delta^2 - \delta)b^2 \geq 0. \qquad (24)$$

If $2 - 2\delta - \delta^2 \geq 0$, $\delta \leq \delta^{**} \approx 0.73$, and we would be done here. Otherwise, LHS of (24) would reach its minimum at $\tilde{\theta} = \frac{b(\delta^2 + 2\delta - 2)}{2}$. (24) becomes

$$-\frac{b^2(1+\delta)(\delta^2 + 2\delta - 2)^2}{4} + (1 + \delta^2 - \delta)b^2 \geq 0$$

$$4(1 + \delta^2 - \delta) - (1+\delta)(\delta^2 + 2\delta - 2)^2 \geq 0,$$

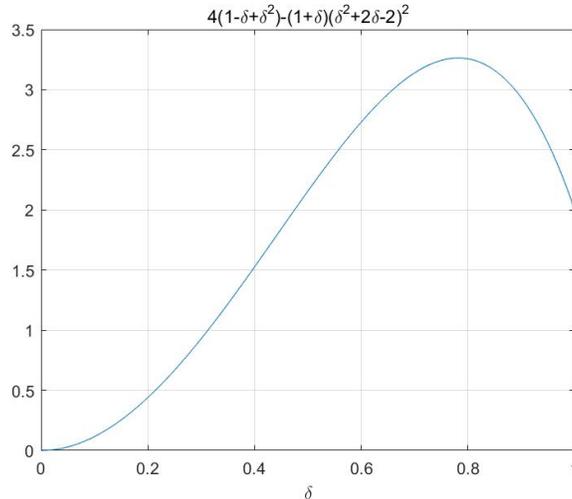which holds not only for $\delta > \delta^{**} \approx 0.73$ but for $\forall \delta \in [0, 1]$.

Figure 18: $4(1 + \delta^2 - \delta) - (1+\delta)(\delta^2 + 2\delta - 2)^2$

Step 4. The above argument only works for $\tilde{\theta} + D \leq 1$. Now we rule out the case of $\tilde{\theta} + D > 1$. If $\tilde{\theta} + D > 1$, the agent with state $\tilde{\theta}$ would deviate to $S(\tilde{\theta}) = 0$ under off-equilibrium belief $\hat{\theta} = 1$ because

$$V(\tilde{\theta}, \tilde{\theta}, \sigma(\tilde{\theta})) = -S^2(\tilde{\theta}) - \delta b^2 < -S_0^2 - \delta b^2 \leq -S_0^2 = -D^2 < -(1 - \tilde{\theta})^2$$
$$\leq -(1 - \tilde{\theta} - b)^2 \leq -\delta(1 - \tilde{\theta} - b)^2 = V(\tilde{\theta}, 1, \tilde{\theta} + b),$$

where the fourth inequality holds since $\tilde{\theta} < \frac{1}{2} - b$ (by the proof of Lemma 8). In addition, the agent will deviate to $S(\tilde{\theta}) = 0$ if off-equilibrium belief $\hat{\theta} = 0$ (by the proof of Lemma 8). Therefore, an agent with state $\tilde{\theta}$ would deviate no matter what the off-equilibrium belief is, which means this is not a perfect Bayesian equilibrium. $\quad\square$

E   Online Appendix: Negative Discriminant Quadratic Games

In this section, we establish the result for quadratic signaling games with $\Delta < 0$. Similarly to the proof of Theorem 2, we only need to consider $k_1 > 0$. We divide the discussion into two cases.

Case 1: $\theta_0 \notin \Theta$

Proposition 7. *In negative discriminant quadratic games, suppose $\theta_0 \notin \Theta$. There exists a unique Riley equilibrium. In this equilibrium, the sender's strategy is nonlinear, continuous, monotonic, and differentiable.*

Proof of Proposition 7. Let $g(\phi, S) \triangleq \ln|S^2 + k_1 S\phi - r\phi^2| - \frac{k_1}{q} \arg\tan(\frac{S}{q\phi} + \frac{k_1}{2q})$. If $\theta_0 > M$, the unique dominant solution crosses $(\theta, S) = (m, 0)$:

$$g(\phi, S^*) = \ln(-r(m - \theta_0)^2) - \frac{k_1}{q}\arg\tan(\frac{k_1}{2q}) \qquad (S^* \geq 0, \ \phi \in [m - \theta_0, M - \theta_0])$$

This satisfies the SOC and is continuous and nonlinear. It is monotonic by $\frac{da_1}{d\theta} = \frac{r\phi}{S^*} > 0$. If $\theta_0 < m$, the unique dominant one crosses $(\theta, S) = (M, 0)$:

$$g(\phi, S^*) = \ln(-r(M - \theta_0)^2) - \frac{k_1}{q}\arg\tan(\frac{k_1}{2q}) \qquad (S^* \leq 0, \ \phi \in [m - \theta_0, M - \theta_0])$$
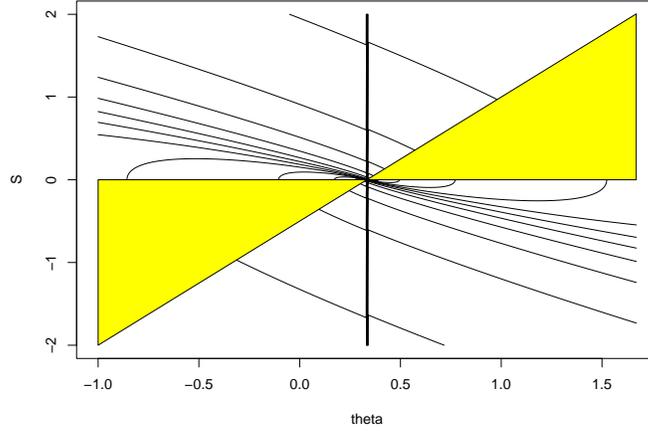
Figure 19: Solutions for $\Delta < 0$
The yellow region is removed by the SOC. This figure is drawn for $k_1 = 1$, $k_2 = 0.4$,
$b_1 = 0.5$, $b_2 = 0.2$, $\delta = 0.5$.

This satisfies the SOC and is continuous and nonlinear. It is monotonic by $\frac{da_1}{d\theta} = \frac{r\phi}{S^*} > 0$. □

Case 2: $\theta_0 \in \Theta$

Proposition 8. *In negative discriminant quadratic games, suppose $\theta_0 \in \Theta$. There is no continuous incentive-compatible separating strategy.*

Proof of Proposition 8. All solutions of (11) feature $S(\theta_0) \neq 0$. By $\frac{da_1}{d\phi} = \frac{r\phi}{S}$, any separating strategy continuous at θ_0 with $S(\theta_0) \neq 0$ must violate separation. □

Thus, an incentive-compatible strategy must jump at θ_0. By proposition 1, it is discontinuous only at θ_0 and jumps upward.

Proposition 9. *In negative discriminant quadratic games, suppose $\theta_0 \in \Theta$. Incentive-compatible separating strategies exist if and only if $k_2 > 0$.*

Proof of Proposition 9. If $k_2 \leq 0$, the SOC implies

$$\frac{\phi}{S} \leq 0.$$

No separating strategy can jump upward. Thus, no incentive-compatible separating strategies exists.

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If $k_2 > 0$, SOC is

$$\frac{S}{\phi} \geq \frac{-k_1(k_2 - 1)}{k_2}$$

by $k_2 < 1$ because $\Delta < 0$. Then there exist discontinuous solutions of (11) jumping upward at θ_0 while satisfying the SOC. To show incentive compatibility, it suffices to show any sender with $\theta \in [m, \theta_0]$ does not mimic $\theta' \in (\theta_0, M]$. By $k_1 > 0$, $k_2 > 0$, $|S(\theta_0^-)| = |S(\theta_0^+)|$. θ mimicking θ' is worse than mimicking $2\theta_0 - \theta'$ for both periods. Incentive-compatible separating strategies thus exist. $\qquad\square$

Among all incentive-compatible separating strategies, the dominant one is closest to $S = 0$ and satisfies

$$S(\theta = m) = \frac{-k_1(k_2 - 1)}{k_2}(m - \theta_0)$$

if $\theta_0 \geq \frac{m+M}{2}$, and it satisfies

$$S(\theta = M) = \frac{-k_1(k_2 - 1)}{k_2}(M - \theta_0)$$

if $\theta_0 < \frac{m+M}{2}$.

From now on, we focus on $k_2 > 0$. $\Delta = k_1^2 + 4\delta(k_2 - 1) < 0$ implies $k_2 < 1 - \frac{k_1^2}{4\delta}$. Thus, $k_2 > 0$ further implies $k_2 \in (0, 1 - \frac{k_1^2}{4\delta})$, which is a very small parameter space. We next show that in most cases, no perfect Bayesian equilibrium exists.

Proposition 10. *In negative discriminant quadratic games, suppose $\theta_0 \in \Theta$ and $k_2 > 0$. If $k_2 < \frac{k_1}{\sqrt{\delta}}$, there is no perfect Bayesian equilibrium.*

Proof of Proposition 10. WLOG, suppose $\theta_0 \geq \frac{m+M}{2}$, and the SOC requires

$$S(\theta = m) \leq \frac{-k_1(k_2 - 1)}{k_2}(m - \theta_0)$$

$$\sigma(\theta = m) \leq k_1 m + b_1 - \frac{k_1(k_2 - 1)}{k_2}(m - \theta_0).$$

By $\Delta < 0, r < 0, \frac{\phi}{S} > 0$, and $\frac{da_1}{d\phi} = \frac{r\phi}{S}$, a_1 is decreasing whenever continuous. Thus,

$$\sigma(\theta = \theta_0) < \sigma(\theta = m) \leq k_1 m + b_1 - \frac{k_1(k_2 - 1)}{k_2}(m - \theta_0),$$

$$|S(\theta = \theta_0)| > k_1(\theta_0 - m) + \frac{k_1(k_2 - 1)}{k_2}(m - \theta_0) = \frac{k_1}{k_2}(\theta_0 - m).$$

If the sender with $\theta = \theta_0$ deviates to $a'_1 = k_1\theta_0 + b_1$, the first-period gain is at least $[\frac{k_1}{k_2}(\theta - m)]^2$. Yet the largest second-period punishment is $\delta(\theta_0 - m)^2$. Therefore, if $(\frac{k_1}{k_2})^2 > \delta$, the sender with $\theta = \theta_0$ deviates for sure and there is no perfect Bayesian equilibrium. $\qquad\square$